Theorem If $X \sim N(\mu, \sigma^2)$ then the random variable $Y = \frac{X-\mu}{\sigma} \sim N(0, 1)$.

Proof Let the random variable X have the normal distribution with probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty.$$

The transformation $Y = g(X) = (X - \mu)/\sigma$ is a 1–1 transformation from $\mathcal{X} = \{x \mid -\infty < x < \infty\}$ to $\mathcal{Y} = \{y \mid y > 0\}$ with inverse $X = g^{-1}(Y) = \mu + \sigma Y$ and Jacobian

$$\frac{dX}{dY} = \sigma.$$

Using the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{dx}{dy}\right| \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\mu+\sigma y-\mu}{\sigma}\right)^2} |\sigma| \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \qquad -\infty < y < \infty, \end{aligned}$$

which is the probability density function of a $N(0,1)$ random variable.

APPL verification: The APPL statements

```
X := NormalRV(mu, sigma):
g := [[x -> (x - mu) / sigma], [-infinity,infinity]]:
Y := Transform(X, g);
```

yield the probability density function of a $N(0,1)$ random variable.