Theorem [UNDER CONSTRUCTION!] If $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, \ldots, n$ are mutually independent and identically distributed random variables, then $Y = \sum_{i=1}^{n} X_i^2/\sigma^2$ has the noncentral chi-square distribution.

Proof [UNDER CONSTRUCTION!] Let $X_i, i = 1, 2, \ldots, n$ have the $N(\mu, \sigma^2)$ distribution with probability density function

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}\sigma} \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty.$$

The transformation $Y_i = g(X_i) = X_i/\sigma$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, -\infty < x < \infty\}$ to $\mathcal{Y} = \{y \,|\, -\infty < y < \infty\}$ with inverse $X_i = g^{-1}(Y_i) = \sigma Y_i$ and Jacobian

$$\frac{dX_i}{dY_i} = \sigma.$$

Using the transformation technique, the probability density function of Y_i is

$$\begin{aligned} f_{Y_i}(y) &= f_{X_i}\left(g^{-1}(y)\right) \left|\frac{dx}{dy}\right| \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\sigma y-\mu}{\sigma}\right)^2} |\sigma| \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\mu/\sigma)^2} \qquad -\infty < y < \infty. \end{aligned}$$

Therefore, $Y_i \sim N(\mu/\sigma, 1)$. Let $V_i = h(Y_i) = Y_i^2$. This is a 2–1 transformation from $\mathcal{Y} = \{y \,|\, -\infty < y < \infty\}$ to $\mathcal{V} = \{v \,|\, v > 0\}$. The domain of the transformation can be divided into $\mathcal{Y}_1 = \{y \,|\, y \le 0\}$ and $\mathcal{Y}_2 = \{y \,|\, y > 0\}$, such that the mapping from $\mathcal{Y}_1$ to $\mathcal{V}$ and $\mathcal{Y}_2$ to $\mathcal{V}$ are each 1–1. The inverse functions are $Y_i = h_1^{-1}(V_i) = -\sqrt{V_i}$ and $Y_i = h_2^{-1}(V_i) = \sqrt{V_i}$, and the Jacobians are

$$J_1 = \frac{dY_i}{dV_i} = -\frac{1}{2\sqrt{V_i}}$$

and

$$J_2 = \frac{dY_i}{dV_i} = \frac{1}{2\sqrt{V_i}}.$$

Using the transformation technique, the probability density function of V_i is

$$\begin{aligned} f_{V_i}(v) &= f_{Y_i}\left(h_1^{-1}(v)\right)|J_1| + f_{Y_i}\left(h_2^{-1}(v)\right)|J_2| \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(-\sqrt{v}-\mu/\sigma\right)^2} \left|-\frac{1}{2\sqrt{v}}\right| + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\sqrt{v}-\mu/\sigma\right)^2} \left|\frac{1}{2\sqrt{v}}\right| \\ &= \frac{1}{2\sqrt{2\pi v}} \left(e^{-\frac{1}{2}\left(-\sqrt{v}-\mu/\sigma\right)^2} + e^{-\frac{1}{2}\left(\sqrt{v}-\mu/\sigma\right)^2}\right) \qquad v_i > 0. \end{aligned}$$

This proof is not complete. The result is given on page 75 of Forbes, Evans, Hastings, and Peacock (Statistical Distributions, fourth edition, John Wiley and Sons, 2011).