Theorem The exponentiation of a $N(\mu,\, \sigma^2)$ random variable is a log normal$(\alpha,\, \beta)$ random variable.

Proof Let the random variable X have the normal distribution with probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad\qquad -\infty < x < \infty.$$

The transformation $Y = g(X) = e^X$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, -\infty < x < \infty\}$ to $\mathcal{Y} = \{y \,|\, y > 0\}$ with inverse $X = g^{-1}(Y) = \ln(Y)$ and Jacobian

$$\frac{dX}{dY} = \frac{1}{Y}.$$

Therefore by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right) \left|\frac{dx}{dy}\right| \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right) e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2} \left|\frac{1}{y}\right| \qquad\qquad y > 0. \end{aligned}$$

Let $\mu = \ln(\alpha)$ and $\sigma = \beta$. Then

$$f_Y(y) = \left(\frac{1}{\sqrt{2\pi}y\beta}\right) e^{-\frac{1}{2}\left(\frac{\ln(y/\alpha)}{\beta}\right)^2} \qquad\qquad y > 0,$$

which is the probability density function of the log normal distribution.

APPL Verification: The APPL statements

```
X := NormalRV(mu, sigma);
g := [[x -> exp(x)],[-infinity, infinity]];
Y := Transform(X, g);
Z := LogNormalRV(mu, sigma);
```

yield identical probability density functions for `Y` and `Z`.