Theorem If $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$ and $X_1, X_2, \ldots, X_n$ are mutually independent, then

$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

for nonzero real constants $a_1, a_2, \ldots, a_n$.

Proof The moment generating function of X_i is

$$M_{X_i}(t) = \exp\left(\mu_i t + \frac{1}{2}\,\sigma_i^2 t^2\right) \qquad -\infty < t < \infty;\ i = 1, \ldots, n.$$

The moment generating function of $a_i X_i$ is

$$M_{a_i X_i}(t) = \exp\left(a_i \mu_i t + \frac{1}{2}\,a_i^2 \sigma_i^2 t^2\right) \qquad -\infty < t < \infty;\ i = 1, \ldots, n.$$

Since $X_1, X_2, \ldots, X_n$ are mutually independent, the moment generating function of the linear combination is

$$M_{a_1 X_1 + a_2 X_2 + \cdots + a_n X_n}(t) = \prod_{i=1}^{n} M_{a_i x_i}(t) = \exp\left(\left(\sum_{i=1}^{n} a_i \mu_i\right) t + \frac{1}{2}\left(\sum_{i=1}^{n} a_i^2 \sigma_i^2\right) t^2\right)$$

for $-\infty < t < \infty$. Therefore

$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

APPL illustration: For $n = 3$ and $a_1 = a_2 = a_3 = 1$, the APPL statements

```
X1 := NormalRV(mu1, sigma1);
X2 := NormalRV(mu2, sigma2);
X3 := NormalRV(mu3, sigma3);
Convolution(X1, X2, X3);
```

yield the probability density function of a $N(\mu_1 + \mu_2 + \mu_3,\ \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ random variable.