Theorem If $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, \ldots, n$ are mutually independent and identically distributed random variables, then $Y = \sum_{i=1}^{n}((X_i - \mu)/\sigma)^2$ has the chi-square distribution with n degrees of freedom.

Proof Let $X_i, i = 1, 2, \ldots, n$ have the $N(\mu, \sigma^2)$ distribution with probability density function

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty.$$

The transformation $Y_i = g(X_i) = (X_i - \mu)/\sigma$ is a 1–1 transformation from $\mathcal{X} = \{x \mid -\infty < x < \infty\}$ to $\mathcal{Y} = \{y \mid -\infty < y < \infty\}$ with inverse $X_i = g^{-1}(Y_i) = \mu + \sigma Y_i$ and Jacobian

$$\frac{dX_i}{dY_i} = \sigma.$$

Using the transformation technique, the probability density function of Y_i is

$$\begin{aligned} f_{Y_i}(y) &= f_{X_i}\left(g^{-1}(y)\right)\left|\frac{dx}{dy}\right| \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\mu+\sigma y-\mu}{\sigma}\right)^2} |\sigma| \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \qquad -\infty < y < \infty. \end{aligned}$$

Let $V_i = Y_i^2$. The cumulative distribution function of V_i is

$$\begin{aligned} F_V(v) &= P(V_i \le v) \\ &= P(Y_i^2 \le v) \\ &= P(-\sqrt{v} \le Y_i \le \sqrt{v}) \\ &= 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}}\, e^{-v^2/2} dv \qquad -\infty < v < \infty \end{aligned}$$

by the symmetry of the standard normal distribution around 0. Letting $u = v^2$,

$$\begin{aligned} F_V(u) &= 2\int_0^{u} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}\left(\frac{1}{2\sqrt{u}}\right) du \\ &= \int_0^{u} \frac{1}{\sqrt{\pi}\sqrt{2}}\, u^{1/2-1} e^{-u/2} du \qquad u > 0. \end{aligned}$$

Taking the derivative with respect to u,

$$f_V(u) = \frac{1}{\Gamma(1/2)\, 2^{1/2}}\, u^{1/2-1} e^{-u/2} \qquad u > 0,$$

the probability density function of the chi-square distribution with 1 degree of freedom. Because $V_i^2 \sim \chi^2_{(1)}, i = 1, 2, \ldots, n$, the moment generating function of V_i is

$$M_{V_i}(t) = (1 - 2t)^{-1/2} \qquad t < 1/2.$$

Because the V_i are mutually independent, the moment generating function of $Z = \sum_{i=1}^{n} V_i^2$ is

$$
\begin{aligned}
M_Z(t) &= \prod_{i=1}^{n} M_{V_i}(t) \\
&= \prod_{i=1}^{n} (1-2t)^{-1/2} \\
&= (1-2t)^{-n/2} \qquad\qquad t < 1/2,
\end{aligned}
$$

the moment generating function of a chi-square random variable with n degrees of freedom.

APPL illustration: The APPL statements

```
Y := NormalRV(0, 1);
g := [[x -> x ^ 2, x -> x ^ 2], [-infinity, 0, infinity]];
Z := Transform(Y, g);
Y := ConvolutionIID(Z, 3);
ChiSquareRV(3);
```

illustrate the result above for $n = 3$.