Theorem If $X_i \sim N(\mu, \sigma^2), i = 1, 2, \ldots, n$ are mutually independent and identically distributed random variables, then $Y = \sum_{i=1}^{n} ((X_i - \mu)/\sigma)^2$ has the chi-square distribution with $n$ degrees of freedom.

Proof Let $X_i, i = 1, 2, \ldots, n$ have the $N(\mu, \sigma^2)$ distribution with probability density function

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, -\infty < x < \infty.$$  

The transformation $Y_i = g(X_i) = (X_i - \mu)/\sigma$ is a 1–1 transformation from $X = \{x \mid -\infty < x < \infty\}$ to $Y = \{y \mid -\infty < y < \infty\}$ with inverse $X_i = g^{-1}(Y_i) = \mu + \sigma Y_i$ and Jacobian

$$\frac{dX_i}{dY_i} = \sigma.$$

Using the transformation technique, the probability density function of $Y_i$ is

$$f_{Y_i}(y) = f_{X_i}(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{\mu+\sigma y - \mu}{\sigma})^2} |\sigma| = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, -\infty < y < \infty.$$

Let $V_i = Y_i^2$. The cumulative distribution function of $V_i$ is

$$F_V(v) = P(V_i \leq v) = P(Y_i^2 \leq v) = P(-\sqrt{v} \leq Y_i \leq \sqrt{v}) = 2 \int_{0}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, -\infty < v < \infty$$

by the symmetry of the standard normal distribution around 0. Letting $u = v^2$,

$$F_V(u) = 2 \int_{0}^{u} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \left( \frac{1}{2\sqrt{u}} \right) du = \int_{0}^{u} \frac{1}{\sqrt{\pi}u^{1/2}} u^{1/2-1} e^{-u/2} du, u > 0.$$

Taking the derivative with respect to $u$,

$$f_V(u) = \frac{1}{\Gamma(1/2) 2^{1/2}} u^{1/2-1} e^{-u/2}, u > 0,$$

the probability density function of the chi-square distribution with 1 degree of freedom. Because $V_i^2 \sim \chi^2(1), i = 1, 2, \ldots, n$, the moment generating function of $V_i$ is

$$M_{V_i}(t) = (1 - 2t)^{-1/2}, t < 1/2.$$
Because the $V_i$ are mutually independent, the moment generating function of $Z = \sum_{i=1}^{n} V_i^2$ is

$$M_Z(t) = \prod_{i=1}^{n} M_{V_i}(t)$$

$$= \prod_{i=1}^{n} (1 - 2t)^{-1/2}$$

$$= (1 - 2t)^{-n/2} \quad t < 1/2,$$

the moment generating function of a chi-square random variable with $n$ degrees of freedom.

**APPL illustration:** The APPL statements

```
Y := NormalRV(0, 1);
g := [[x -> x ^ 2, x -> x ^ 2], [-infinity, 0, infinity]];
Z := Transform(Y, g);
Y := ConvolutionIID(Z, 3);
ChiSquareRV(3);
```

illustrate the result above for $n = 3$. 

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