Theorem As $\kappa \to 0$ for a Muth random variable with parameter κ, the limiting distribution is exponential with mean 1.

Proof A Muth random variable X, with parameter κ has probability density function

$$f_X(x) = (e^{\kappa x} - \kappa)\, e^{[(-1/\kappa)e^{\kappa x} + \kappa x + 1/\kappa]} \qquad x > 0.$$

Now we want the limit as $\kappa \to 0$. First consider the second factor in the product

$$e^{[(-1/\kappa)e^{\kappa x} + \kappa x + 1/\kappa]}.$$

Because $\exp(\cdot)$ is everywhere continuous, we can consider the limit as $\kappa \to 0$ of the exponent. Using the Taylor series expansion of $e^{\kappa x}$ about $x = 0$, for this exponent,

$$\begin{aligned}
-\frac{1}{\kappa}e^{\kappa x} + \kappa x + \frac{1}{\kappa} &= -\frac{1}{\kappa}\left(-1 + e^{\kappa x}\right) + \kappa x \\
&= -\frac{1}{\kappa}\left(-1 + \left[1 + \kappa x + \frac{\kappa^2 x^2}{2!} + \frac{\kappa^3 x^3}{3!} + \cdots\right]\right) + \kappa x \\
&= \left(-x - \frac{\kappa x^2}{2!} - \frac{\kappa^2 x^3}{3!} - \frac{\kappa^3 x^4}{4!} - \cdots\right) + \kappa x \\
&= -x(1-\kappa) + \left(-\frac{\kappa x^2}{2!} - \frac{\kappa^2 x^3}{3!} - \frac{\kappa^3 x^4}{4!} - \cdots\right).
\end{aligned}$$

The limit of the above expression as $\kappa \to 0$ is $-x$. So, it follows that

$$\lim_{\kappa \to 0} e^{[(-1/\kappa)e^{\kappa x} + \kappa x + 1/\kappa]} = e^{-x}.$$

Returning to the first factor in the original product,

$$\lim_{\kappa \to 0}(e^{\kappa x} - \kappa) = 1.$$

Combining both parts,

$$\begin{aligned}
\lim_{\kappa \to 0}(e^{\kappa x} - \kappa)e^{[(-1/\kappa)e^{\kappa x} + \kappa x + 1/\kappa]} &= \lim_{\kappa \to 0}(e^{\kappa x} - \kappa) \cdot \lim_{\kappa \to 0} e^{[(-1/\kappa)e^{\kappa x} + \kappa x + (1/\kappa)]} \\
&= 1 \cdot e^{-x} \\
&= e^{-x},
\end{aligned}$$

which is the probability density function of an exponential random variable with mean 1.

APPL verification: The APPL statements

```
X := MuthRV(kappa);
limit(X[1][1](x), kappa = 0);
Y := ExponentialRV(1);
```

verify the result. Letting $\kappa \to 0$ and the `ExponentialRV(1)` command both give a probability density function for an exponential random variable with mean 1.