Theorem The product of n mutually independent log normal random variables is log normal.

Proof (an elegant proof that uses the L property of the normal distribution) Let $X_1 \sim \text{lognormal}\,(\mu_1, \sigma_1^2)$ and $X_2 \sim \text{lognormal}\,(\mu_2, \sigma_2^2)$ (using the parametrization in the chart, $\mu = \ln\alpha$ and $\beta = \sigma$) be independent. Then $X_1 = e^{X_1'}$ and $X_2 = e^{X_2'}$ for independent normally distributed random variables $X_1' \sim N(\mu_1, \sigma_1^2)$ and $X_2' \sim N(\mu_2, \sigma_2^2)$. We have $Y = X_1X_2 = e^{X_1'+X_2'}$ and the result for $n = 2$ follows from the linear combination property (L) of normal distribution. The general result follows by induction.

Proof [A brutish, unfinished proof that remains ... UNDER CONSTRUCTION] Let $X_1 \sim \log\text{normal}(\alpha_1, \beta_1)$ and $X_2 \sim \log\text{normal}(\alpha_2, \beta_2)$ be independent log normal random variables. We can write their probability density functions as

$$f_{X_1}(x_1) = \frac{1}{x_1\beta_1\sqrt{2\pi}} e^{-\frac{1}{2}(\ln(x_1/\alpha_1)/\beta_1)^2} \qquad x_1 > 0$$

and

$$f_{X_2}(x_2) = \frac{1}{x_2\beta_2\sqrt{2\pi}} e^{-\frac{1}{2}(\ln(x_2/\alpha_2)/\beta_2)^2} \qquad x_2 > 0.$$

Since X_1 and X_2 are independent, the joint probability density function of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi x_1 x_2 \beta_1\beta_2} e^{-\frac{1}{2}\left[(\ln(x_1/\alpha_1)/\beta_1)^2 + (\ln(x_2/\alpha_2)/\beta_2)^2\right]} \qquad x_1 > 0, x_2 > 0.$$

Consider the 2×2 transformation

$$Y_1 = g_1(X_1, X_2) = X_1X_2 \qquad \text{and} \qquad Y_2 = g_2(X_1, X_2) = X_2$$

which is a 1–1 transformation from $\mathcal{X} = \{(x_1, x_2)\,|\,x_1 > 0, x_2 > 0\}$ to $\mathcal{Y} = \{(y_1, y_2)\,|\,y_1 > 0, y_2 > 0\}$ with inverses

$$X_1 = g_1^{-1}(Y_1, Y_2) = \frac{Y_1}{Y_2} \qquad \text{and} \qquad X_2 = g_2^{-1}(Y_1, Y_2) = Y_2$$

and Jacobian

$$J = \begin{vmatrix} \frac{1}{Y_2} & -\frac{Y_1}{Y_2^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{Y_2}.$$

Therefore by the transformation technique, the joint probability density function of Y_1 and Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}\left(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\right)|J| \\ &= \frac{1}{2\pi y_1\beta_1\beta_2} e^{-\frac{1}{2}\left[(\ln(y_1/\alpha_1 y_2)/\beta_1)^2 + (\ln(y_2/\alpha_2)/\beta_2)^2\right]} \left|\frac{1}{y_2}\right| \end{aligned}$$

for $y_1 > 0, y_2 > 0$. The probability density function of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^\infty f_{Y_1,Y_2}(y_1, y_2)\, dy_2 \\ &= \int_0^\infty \frac{1}{2\pi y_1 y_2 \beta_1 \beta_2} e^{-\frac{1}{2}\left[(\ln(y_1/\alpha_1 y_2)/\beta_1)^2 + (\ln(y_2/\alpha_2)/\beta_2)^2\right]}\, dy_2 \\ &= \end{aligned}$$

which is the probability density function of a log normal random variable. This integral can be computed in Maple with the statements below.

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assume(alpha1 > 0);
assume(alpha2 > 0);
assume(beta1 > 0);
assume(beta2 > 0);
int(exp(-((log(y1/(alpha1 * y2)) / beta1) ^ 2 +
         (log(y2/alpha2) / beta2) ^ 2) / 2) /
         (2 * Pi * y1 * y2 * beta1 * beta2), y2 = 0 .. infinity);
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Induction can be used with the above result to verify that the product of n mutually independent log normal random variables is log normal. Let $X_1, X_2, \ldots, X_n$ be n mutually independent log normal random variables. Consider their product $X_1 X_2 \ldots X_n$. By the result above, $X_1 X_2$ is log normal. Suppose we've demonstrated that $\prod_{i=1}^{k} X_i$ is a log normal random variable. Consider $\prod_{i=1}^{k+1} X_i$. Since X_{k+1} is also log normal, $\prod_{i=1}^{k+1} X_i$ is log normal by the result above. It follows by induction that $X_1 X_2 \ldots X_n$ must be log normal.