Theorem The product of \( n \) mutually independent log normal random variables is log normal.

Proof (an elegant proof that uses the L property of the normal distribution) Let \( X_1 \sim \text{lognormal} (\mu_1, \sigma_1^2) \) and \( X_2 \sim \text{lognormal} (\mu_2, \sigma_2^2) \) (using the parametrization in the chart, \( \mu = \ln \alpha \) and \( \beta = \sigma \)) be independent. Then \( X_1 = e^{X'_1} \) and \( X_2 = e^{X'_2} \) for independent normally distributed random variables \( X'_1 \sim N(\mu_1, \sigma_1^2) \) and \( X'_2 \sim N(\mu_2, \sigma_2^2) \). We have \( Y = X_1 X_2 = e^{X'_1 + X'_2} \) and the result for \( n = 2 \) follows from the linear combination property (L) of normal distribution. The general result follows by induction.

Proof [A brutish, unfinished proof that remains ... UNDER CONSTRUCTION] Let \( X_1 \sim \text{log normal}(\alpha_1, \beta_1) \) and \( X_2 \sim \text{log normal}(\alpha_2, \beta_2) \) be independent log normal random variables. We can write their probability density functions as

\[
f_{X_1}(x_1) = \frac{1}{x_1 \beta_1 \sqrt{2\pi}} e^{-\frac{1}{2} \left(\ln(x_1/\alpha_1)/\beta_1\right)^2} \quad x_1 > 0
\]

and

\[
f_{X_2}(x_2) = \frac{1}{x_2 \beta_2 \sqrt{2\pi}} e^{-\frac{1}{2} \left(\ln(x_2/\alpha_2)/\beta_2\right)^2} \quad x_2 > 0.
\]

Since \( X_1 \) and \( X_2 \) are independent, the joint probability density function of \( X_1 \) and \( X_2 \) is

\[
f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi x_1 x_2 \beta_1 \beta_2} e^{-\frac{1}{2} \left[\ln(x_1/\alpha_1)/\beta_1\right]^2 + \left(\ln(x_2/\alpha_2)/\beta_2\right)^2} \quad x_1 > 0, x_2 > 0.
\]

Consider the \( 2 \times 2 \) transformation

\[
Y_1 = g_1(X_1, X_2) = X_1 X_2 \quad \text{and} \quad Y_2 = g_2(X_1, X_2) = X_2
\]

which is a 1–1 transformation from \( \mathcal{X} = \{(x_1, x_2) \mid x_1 > 0, x_2 > 0\} \) to \( \mathcal{Y} = \{(y_1, y_2) \mid y_1 > 0, y_2 > 0\} \) with inverses

\[
X_1 = g_1^{-1}(Y_1, Y_2) = \frac{Y_1}{Y_2} \quad \text{and} \quad X_2 = g_2^{-1}(Y_1, Y_2) = Y_2
\]

and Jacobian

\[
J = \begin{vmatrix}
\frac{1}{Y_2} & -\frac{Y_1}{Y_2^2} \\
0 & 1
\end{vmatrix} = \frac{1}{Y_2}.
\]

Therefore by the transformation technique, the joint probability density function of \( Y_1 \) and \( Y_2 \) is

\[
f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}\left(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\right) |J| = \frac{1}{2\pi y_1 \beta_1 \beta_2} e^{-\frac{1}{2} \left[\ln(y_1/\alpha_1 y_2)/\beta_1\right]^2 + \left(\ln(y_2/\alpha_2)/\beta_2\right)^2} \quad \frac{1}{Y_2}
\]
for $y_1 > 0, y_2 > 0$. The probability density function of $Y_1$ is

$$f_{Y_1}(y_1) = \int_0^\infty f_{Y_1,Y_2}(y_1, y_2) dy_2$$

$$= \int_0^\infty \frac{1}{2\pi y_1 y_2 \beta_1 \beta_2} e^{-\frac{1}{2}[\frac{\ln(y_1/\alpha_1 y_2)}{\beta_1}^2 + \frac{\ln(y_2/\alpha_2)}{\beta_2}^2]} dy_2$$

which is the probability density function of a log normal random variable. This integral can be computed in Maple with the statements below.

```maple
assume(alpha1 > 0);
assume(alpha2 > 0);
assume(beta1 > 0);
assume(beta2 > 0);
int(exp(-((log(y1/(alpha1 * y2)) / beta1) ^ 2 +
       (log(y2/alpha2) / beta2) ^ 2) / 2) /
    (2 * Pi * y1 * y2 * beta1 * beta2), y2 = 0 .. infinity);
```

Induction can be used with the above result to verify that the product of $n$ mutually independent log normal random variables is log normal. Let $X_1, X_2, \ldots, X_n$ be $n$ mutually independent log normal random variables. Consider their product $X_1 X_2 \ldots X_n$. By the result above, $X_1 X_2$ is log normal. Suppose we’ve demonstrated that $\prod_{i=1}^k X_i$ is a log normal random variable. Consider $\prod_{i=1}^{k+1} X_i$. Since $X_{k+1}$ is also log normal, $\prod_{i=1}^{k+1} X_i$ is log normal by the result above. It follows by induction that $X_1 X_2 \ldots X_n$ must be log normal.