Theorem The natural logarithm of a log logistic(λ, κ) random variable is a logistic(λ, κ) random variable.

Proof Let the random variable X have the log logistic distribution with probability density function

$$f_X(x) = \frac{\lambda\kappa(\lambda x)^{\kappa-1}}{[1+(\lambda x)^\kappa]^2} \qquad x > 0.$$

The transformation $Y = g(X) = \ln X$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, x > 0\}$ to $\mathcal{Y} = \{y \,|\, -\infty < y < \infty\}$ with inverse $X = g^{-1}(Y) = e^Y$ and Jacobian

$$\frac{dX}{dY} = e^Y.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right| \\ &= \frac{\lambda\kappa\left(\lambda e^y\right)^{\kappa-1}}{[1+(\lambda e^y)^\kappa]^2} \left|e^y\right| \\ &= \frac{\lambda^\kappa \kappa e^{y\kappa}}{[1+(\lambda e^y)^\kappa]^2} \qquad -\infty < y < \infty, \end{aligned}$$

which is the probability density function of the logistic distribution.

APPL verification: The APPL statements

```
X := LogLogisticRV(lambda, kappa);
g := [[x -> ln(x)], [0, infinity]];
Y := Transform(X, g);
Z := LogisticRV(lambda, kappa);
```

yield identical the functional forms

$$f_Y(y) = \frac{\lambda^\kappa \kappa e^{y\kappa}}{[1+(\lambda e^y)^\kappa]^2} \qquad -\infty < y < \infty$$

for the random variables Y and Z, which verifies that the natural logarithm of a log-logistic random variable has the logistic distribution.