Theorem [UNDER CONSTRUCTION!] If $X \sim$ inverse Gaussian(λ, μ), then

$$Y = \lambda(X-\mu)^2/(\mu^2 X)$$

has the chi-square distribution.

Proof 1 [UNDER CONSTRUCTION!] Let the random variable X have the inverse Gaussian distribution with probability density function

$$f(x) = \sqrt{\frac{\lambda}{2\pi\, x^3}}\, e^{-\frac{\lambda(x-\mu)^2}{2x\mu^2}} \qquad x > 0.$$

The transformation $Y = g(X) = \lambda(X-\mu)^2/(\mu^2 X)$ is a 1–1 transformation from $\mathcal{X} = \{x \mid x > 0\}$ to $\mathcal{Y} = \{y \mid y > 0\}$ with inverse $X = g^{-1}(Y) = \mu(\mu y + 2\lambda + \sqrt{y^2\mu^2 + 4y\mu\lambda}\,)/2\lambda$ (by using the quadratic formula) and Jacobian

$$\frac{dX}{dY} = \frac{\mu}{2\lambda}\left(\mu + \frac{Y\mu^2 + 2\mu\lambda}{\sqrt{Y^2\mu^2 + 4Y\mu\lambda}}\right).$$

Therefore by the transformation technique, the probability density function of Y is

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| \\
&= \sqrt{\frac{\lambda}{2\pi\left(\mu(\mu y + 2\lambda + \sqrt{y^2\mu^2 + 4y\mu\lambda}\,)/2\lambda\right)^3}} \times \\
&\quad \exp\left(\frac{-\lambda((\mu(\mu y + 2\lambda + \sqrt{y^2\mu^2 + 4y\mu\lambda}\,)/2\lambda) - \mu)^2}{2(\mu(\mu y + 2\lambda + \sqrt{y^2\mu^2 + 4y\mu\lambda}\,)/2\lambda)\mu^2}\right) \times \\
&\quad \left|\frac{\mu}{2\lambda}\left(\mu + \frac{y\mu^2 + 2\mu\lambda}{\sqrt{y^2\mu^2 + 4y\mu\lambda}}\right)\right| \qquad y > 0.
\end{aligned}$$

This should reduce to a chi-square random variable with one degree of freedom according to Chhikara and Folks (The Inverse Gaussian Distribution: Theory, Methodology, and Applications, 1989, Marcel Dekker, Inc., pages 15 and 53).

Proof 2 [UNDER CONSTRUCTION!] Part of this proof is from Seshadri, *The Inverse Gaussian Distribution: A Case Study in Exponential Families*, Oxford Science Publishers, 1993, page 83. Let the random variable X have the inverse Gaussian distribution with probability density function

$$f(x) = \sqrt{\frac{\lambda}{2\pi\, x^3}}\, e^{-\frac{\lambda(x-\mu)^2}{2x\mu^2}} \qquad x > 0.$$

First, consider the transformation $Y = \sqrt{\lambda}(X-\mu)/(\mu\sqrt{X})$. This is a 1–1 transformation from $\mathcal{X} = \{x \mid x > 0\}$ to $\mathcal{Y} = \{y \mid -\infty < y < \infty\}$ with inverse $X = g^{-1}(Y) = \mu(\mu Y^2 + 4\lambda + Y\sqrt{Y^2\mu^2 + 4Y\mu}\,)/2\lambda$ and Jacobian

$$\frac{dX}{dY} = \frac{2\mu}{\sqrt{\lambda}} \cdot \frac{Y^{3/2}}{(Y+\mu)}.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\ &= \left(1 - \frac{y}{\sqrt{y^2 + 4\lambda/\mu}}\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \qquad\qquad -\infty < y < \infty. \end{aligned}$$

Now, let $Z = h(Y) = Y^2$. This is a 1–1 transformation from $\mathcal{Y} = \{y \mid -\infty < y < \infty\}$ to $\mathcal{Z} = \{z \mid z > 0\}$ with inverse $Y = g^{-1}(z) = \sqrt{Z}$ and Jacobian

$$\frac{dY}{dZ} = \frac{1}{2\sqrt{Z}}.$$

Therefore by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Z(z) &= f_Y(g^{-1}(z)) \left| \frac{dy}{dz} \right| \\ &= \left(1 - \frac{\sqrt{z}}{\sqrt{z + 4\lambda/\mu}}\right) \frac{1}{\sqrt{2\pi}} e^{-z/2} \left| \frac{1}{2\sqrt{z}} \right| \\ &= \frac{1}{2}\left(1 - \sqrt{\frac{z}{z + 4\lambda/\mu}}\right) \cdot \frac{z^{-1/2} e^{-z/2}}{2^{1/2}\Gamma(1/2)} \qquad\qquad z > 0. \end{aligned}$$

The portion on the right of this expression is the probability density function of a chi-square random variable with one degree of freedom, but the portion on the left of the expression does not vanish.