Theorem The binomial(n, p) distribution is the limit of the hypergeometric(n_1, n_2, n_3) distribution with $p = n_1/n_3$, as $n_3 \to \infty$.

Proof Let the random variable X have the hypergeometric(n_1, n_2, n_3) distribution. The probability mass function of X is

$$\begin{aligned}
f(x) &= \binom{n_1}{x}\binom{n_3-n_1}{n_2-x}\Big/\binom{n_3}{n_2} \\
&= \frac{n_1!}{x!(n_1-x)!}\,\frac{(n_3-n_1)!}{(n_2-x)![(n_3-n_1)-(n_2-x)]!}\,\frac{n_2!(n_3-n_2)!}{n_3!} \\
&= \frac{n_2!}{x!(n_2-x)!}\,\frac{n_1!(n_3-n_1)!(n_3-n_2)!}{(n_1-x)!(n_3-n_1-n_2+x)!n_3!} \\
&= \binom{n_2}{x}\frac{[n_1(n_1-1)\ldots(n_1-x+1)][(n_3-n_1)(n_3-n_1-1)\ldots(n_3-n_1-n_2+x+1)]}{n_3(n_3-1)\ldots(n_3-n_2+1)}
\end{aligned}$$

for $x = 0, 1, 2, \ldots, n_2$. It must be the case that $n_1 \to \infty$ because $n_1 = pn_3$ and $n_3 \to \infty$. We expect that $n_3 - n_1 \leqslant n_3 - n_2$ and n_2 could be ignored as n_1, n_3 go to infinity. Set $q = 1/p = n_3/n_1$ then

$$\begin{aligned}
f(x) &= \binom{n_2}{x}\frac{[n_1(n_1-1)\ldots(n_1-x+1)][(q-1)n_1((q-1)n_1-1)\ldots((q-1)n_1-n_2+x+1)]}{qn_1(qn_1-1)\ldots(qn_1-n_2+1)} \\
&= \binom{n_2}{x}\left(\frac{1}{q}\right)^x\left(\frac{q-1}{q}\right)^{n_2-x} \\
&= \binom{n_2}{x}p^x(1-p)^{n_2-x} \qquad\qquad x = 0, 1, 2, \ldots, n_2,
\end{aligned}$$

which is the probability mass function for the binomial(n_2, p) distribution.