Theorem If X_1, X_2, ..., X_n are mutually independent and identically distributed random variables from a geometric(p) population, then $X_1 + X_2 + \cdots + X_n$ has the Pascal (negative binomial)(n, p) distribution.

Proof The geometric distribution has probability mass function

$$f_X(x) = p(1-p)^x \qquad x = 0, 1, 2, \ldots,$$

which represents the probability of exactly x failures prior to the first success. The associated moment generating function is

$$\begin{aligned} M_X(t) &= E\left[e^{tX}\right] \\ &= \sum_{x=0}^{\infty} e^{tx} p(1-p)^x \\ &= p \sum_{x=0}^{\infty} \left(e^t(1-p)\right)^x \\ &= \frac{p}{1-(1-p)e^t} \qquad t < -\ln(1-p). \end{aligned}$$

Let $Y = X_1 + X_2 + \cdots + X_n$. The moment generating function of Y is

$$M_Y(t) = \left(\frac{p}{1-(1-p)e^t}\right)^n \qquad t < -\ln(1-p).$$

This moment generating function is identified as that of a Pascal(n, p) distribution, which proves the result.

APPL verification: The APPL statements

```
assume(p > 0, p < 1);
assume(n, posint);
X := [[x -> p * (1 - p) ^ x], [0 .. infinity], ["Discrete", "PDF"]];
MGF(X) ^ n;
Y :=  [[y -> (n + y - 1)! * p ^ n * (1 - p) ^ y / (y! * (n - 1)!)],
      [0 .. infinity], ["Discrete", "PDF"]];
MGF(Y);
```

verify that the appropriate moment generating functions are equal.