Theorem The limiting distribution of the gamma(α, β) distribution is the $N(\mu, \sigma^2)$ distribution where $\mu = \alpha\beta$ and $\sigma^2 = \alpha^2\beta$.

Proof Let the random variable X have the gamma(α, β) distribution with probability density function

$$f_X(x) = \frac{x^{\beta-1}\mathrm{e}^{-x/\alpha}}{\alpha^\beta \Gamma(\beta)} \qquad x > 0.$$

The moment generating function of X is

$$M_X(t) = (1 - \alpha t)^{-\beta} \qquad t < \frac{1}{\alpha}.$$

The mean of X is $E[X] = \alpha\beta$ and the variance of X is $V[X] = \alpha^2\beta$. Subtract the mean and divide by the standard deviation before taking the limit. The transformation

$$Y = g(X) = (X - \alpha\beta)\Big/\left(\alpha\sqrt{\beta}\right) = X\Big/\left(\alpha\sqrt{\beta}\right) - \sqrt{\beta}$$

is a 1–1 transformation from $\mathcal{X} = \{x \,|\, x > 0\}$ to $\mathcal{Y} = \{y \,|\, y > -\sqrt{\beta}\}$. The moment generating function of Y is

$$\begin{aligned}
M_Y(t) &= E\left[e^{tY}\right] \\
&= E\left[e^{t\left(X/\left(\alpha\sqrt{\beta}\right) - \sqrt{\beta}\right)}\right] \\
&= e^{-t\sqrt{\beta}} E\left[e^{t\left(X/\left(\alpha\sqrt{\beta}\right)\right)}\right] \\
&= e^{-t\sqrt{\beta}} M_X\left(t\Big/\left(\alpha\sqrt{\beta}\right)\right) \\
&= e^{-t\sqrt{\beta}}\left(1 - \left(t\Big/\sqrt{\beta}\right)\right)^{-\beta} \qquad t < \sqrt{\beta}.
\end{aligned}$$

The limiting moment generating function of Y is

$$\lim_{\beta \to \infty} M_Y(t) = \lim_{\beta \to \infty} e^{-t\sqrt{\beta}}\left(1 - \left(t\Big/\sqrt{\beta}\right)\right)^{-\beta} = e^{t^2/2} \qquad -\infty < t < \infty,$$

which is the moment generating function of a standard normal random variable. This limit can be found using the maple statement

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limit(exp(-t * sqrt(beta)) * (1 - (t / sqrt(beta))) ^ (-beta), beta = infinity);
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which confirms that the limiting distribution of the gamma distribution as $\beta \to \infty$ is the normal distribution.