Theorem  The limiting distribution of the gamma($\alpha, \beta$) distribution is the $N(\mu, \sigma^2)$ distribution where $\mu = \alpha \beta$ and $\sigma^2 = \alpha^2 \beta$.

Proof  Let the random variable $X$ have the gamma($\alpha, \beta$) distribution with probability density function

$$f_X(x) = \frac{x^{\beta-1}e^{-x/\alpha}}{\alpha^\beta \Gamma(\beta)} \quad x > 0.$$ 

The moment generating function of $X$ is

$$M_X(t) = (1 - \alpha t)^{-\beta} \quad t < \frac{1}{\alpha}.$$ 

The mean of $X$ is $E[X] = \alpha \beta$ and the variance of $X$ is $V[X] = \alpha^2 \beta$. Subtract the mean and divide by the standard deviation before taking the limit. The transformation

$$Y = g(X) = \left( X - \alpha \beta \right) / \left( \alpha \sqrt{\beta} \right) = X / \left( \alpha \sqrt{\beta} \right) - \sqrt{\beta}$$

is a 1–1 transformation from $\mathcal{X} = \{ x \mid x > 0 \}$ to $\mathcal{Y} = \{ y \mid y > -\sqrt{\beta} \}$. The moment generating function of $Y$ is

$$M_Y(t) = E \left[ e^{tY} \right] = E \left[ e^{t \left( x / (\alpha \sqrt{\beta}) - \sqrt{\beta} \right)} \right]$$

$$= e^{-t \sqrt{\beta}} E \left[ e^{t \left( x / (\alpha \sqrt{\beta}) \right)} \right]$$

$$= e^{-t \sqrt{\beta}} M_X \left( t / (\alpha \sqrt{\beta}) \right)$$

$$= e^{-t \sqrt{\beta}} \left( 1 - \left( t / \sqrt{\beta} \right) \right)^{-\beta} \quad t < \sqrt{\beta}.$$ 

The limiting moment generating function of $Y$ is

$$\lim_{\beta \to \infty} M_Y(t) = \lim_{\beta \to \infty} e^{-t \sqrt{\beta}} \left( 1 - \left( t / \sqrt{\beta} \right) \right)^{-\beta} = e^{t^2/2} \quad -\infty < t < \infty,$$

which is the moment generating function of a standard normal random variable. This limit can be found using the maple statement

$\text{limit(exp(-t * sqrt(beta)) * (1 - (t / sqrt(beta))) ^ (-beta), beta = infinity);}$

which confirms that the limiting distribution of the gamma distribution as $\beta \to \infty$ is the normal distribution.