Theorem The natural logarithm of a gamma random variable follows the log gamma distribution.

Proof Let the gamma random variable X have probability density function

$$f_X(x) = \frac{1}{\alpha^\beta \Gamma(\beta)} x^{\beta-1} e^{-x/\alpha} \qquad x > 0.$$

The transformation $Y = g(X) = \ln X$ is a 1–1 transformation from $\mathcal{X} = \{x \mid x > 0\}$ to $\mathcal{Y} = \{y \mid -\infty < y < \infty\}$. The inverse of the transformation is $X = g^{-1}(Y) = e^Y$, and the associated Jacobian is $\frac{dX}{dY} = e^Y$. By the transformation theorem, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\ &= \frac{1}{\alpha^\beta \Gamma(\beta)} e^{y(\beta-1)} e^{-e^y/\alpha} \left| e^y \right| \\ &= \frac{1}{\alpha^\beta \Gamma(\beta)} e^{\beta y} e^{-y} e^{-e^y/\alpha} e^y \\ &= \frac{1}{\alpha^\beta \Gamma(\beta)} e^{\beta y} e^{-e^y/\alpha} \qquad -\infty < y < \infty, \end{aligned}$$

which is the probability density function of the log gamma distribution.

APPL verification: The APPL statements

```
assume(alpha > 0);
assume(beta  > 0);
X := [[x -> (1 / (alpha ^ beta * GAMMA(beta))) * x ^ (beta - 1) *
            exp(-x / alpha)], [0, infinity], ["Continuous", "PDF"]];
g := [[x -> ln(x)], [0, infinity]];
Y := Transform(X, g);
Z := LogGammaRV(alpha, beta);
```

give the same probability density function for Y and Z, which verifies that the natural logarithm of a gamma random variable has the log gamma distribution.