Theorem The reciprocal of a gamma(α, β) random variable is an inverted gamma(α, β) random variable.

Proof Let the random variable X have the gamma distribution with probability density function

$$f_X(x) = \frac{1}{\alpha^\beta \Gamma(\beta)} x^{\beta-1} e^{-x/\alpha} \qquad x > 0.$$

The transformation $Y = g(X) = 1/X$ is a 1–1 transformation from $\mathcal{X} = \{x \mid x > 0\}$ to $\mathcal{Y} = \{y \mid y > 0\}$ with inverse $X = g^{-1}(Y) = 1/Y$ and Jacobian

$$\frac{dX}{dY} = -\frac{1}{Y^2}.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\ &= \frac{1}{\alpha^\beta \Gamma(\beta)} \left(\frac{1}{y} \right)^{\beta-1} e^{-1/(\alpha y)} \left| -\frac{1}{y^2} \right| \\ &= \frac{1}{\alpha^\beta \Gamma(\beta)} y^{-\beta-1} e^{-1/(y\alpha)} \qquad y > 0. \end{aligned}$$

Swapping the roles of the two parameters by letting $\alpha = \beta$ and $\beta = \alpha$,

$$f_Y(y) = \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{-\alpha-1} e^{-1/(y\beta)} \qquad y > 0,$$

which is the probability density function of the inverted gamma distribution.

APPL verification: The APPL statements

```
assume(alpha > 0);
assume(beta  > 0);
X := [[x -> (1 / (alpha ^ beta * GAMMA(beta))) * x ^ (beta - 1) *
            exp(-x / alpha)], [0, infinity], ["Continuous", "PDF"]];
g := [[x -> 1 / x],[0, infinity]];
Y := Transform(X, g);
Z := InvertedGammaRV(beta, alpha);
```

yield identical functional forms

$$f_Y(y) = \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{-\alpha-1} e^{-1/(y\beta)} \qquad y > 0$$

for the random variables Y and Z, which verifies that the reciprocal of a gamma random variable has the inverted gamma distribution.