Theorem If X_1 and X_2 are independent gamma$(1, \beta_i)$ random variables, for $i = 1, 2$, then X_1/X_2 has the inverted beta distribution.

Proof Let $X_1 \sim$ gamma$(1, \beta_1)$ and $X_2 \sim$ gamma$(1, \beta_2)$ be independent random variables. We can write their probability density functions as

$$f_{X_1}(x_1) = \frac{x_1^{\beta_1-1}\mathrm{e}^{-x_1}}{\Gamma(\beta_1)} \qquad x > 0$$

and

$$f_{X_2}(x_2) = \frac{x_2^{\beta_2-1}\mathrm{e}^{-x_2}}{\Gamma(\beta_2)} \qquad x > 0.$$

Since X_1 and X_2 are independent, the joint probability density function of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \frac{x_1^{\beta_1-1}x_2^{\beta_2-1}e^{-x_1-x_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \qquad x_1 > 0, x_2 > 0.$$

Consider the 2×2 transformation

$$Y_1 = g_1(X_1, X_2) = \frac{X_1}{X_2} \qquad \text{and} \qquad Y_2 = g_2(X_1, X_2) = X_2,$$

which is a 1–1 transformation from $\mathcal{X} = \{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$ to $\mathcal{Y} = \{(y_1, y_2) \mid y_1 > 0, y_2 > 0\}$ with inverses

$$X_1 = g_1^{-1}(Y_1, Y_2) = Y_1 Y_2 \qquad \text{and} \qquad X_2 = g_2^{-1}(Y_1, Y_2) = Y_2$$

and Jacobian

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = Y_2.$$

Therefore, by the transformation technique, the joint probability density function of Y_1 and Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}\left(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\right)|J| \\ &= \frac{y_1^{\beta_1-1}y_2^{\beta_1-1}y_2^{\beta_2-1}e^{-y_1y_2-y_2}}{\Gamma(\beta_1)\Gamma(\beta_2)}|y_2| \\ &= \frac{y_1^{\beta_1-1}y_2^{\beta_1+\beta_2-1}e^{-y_2(y_1+1)}}{\Gamma(\beta_1)\Gamma(\beta_2)} \qquad y_1 > 0, y_2 > 0. \end{aligned}$$

Using integration by parts, the probability density function of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^\infty f_{Y_1,Y_2}(y_1, y_2)\,dy_2 \\ &= \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_0^\infty y_1^{\beta_1-1}y_2^{\beta_1+\beta_2-1}e^{-y_2(y_1+1)}\,dy_2 \\ &= \frac{y_1^{\beta_1-1}(y_1+1)^{-(\beta_2+\beta_1)}\Gamma(\beta_1+\beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \qquad y_1 > 0, \end{aligned}$$

which is the probability density function of an inverted beta(β_1, β_2) random variable.

APPL verification: The APPL statements

```
X1 := GammaRV(1, beta1);
X2 := GammaRV(1, beta2);
g := [[x -> 1 / x], [0, infinity]];
Y := Transform(X2, g);
Z := Product(X1, Y);
```

confirm the result.