Theorem If $X \sim \text{gamma}(\alpha, \beta)$, then $2X/\alpha \sim \chi^2(n)$, where $n = 2\beta$.

Proof Let the random variable X have the gamma distribution with probability density function

$$f_X(x) = \frac{1}{\alpha^\beta \Gamma(\beta)} x^{\beta-1} e^{-x/\alpha} \qquad x > 0.$$

The transformation $Y = g(X) = 2X/\alpha$ is a 1–1 transformation from $\mathcal{X} = \{x \mid x > 0\}$ to $\mathcal{Y} = \{y \mid y > 0\}$ with inverse $X = g^{-1}(Y) = \alpha Y/2$ and Jacobian

$$\frac{\partial X}{\partial Y} = \frac{\alpha}{2}.$$

By the transformation technique, the probability density function of Y is

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\
&= \frac{1}{\alpha^\beta \Gamma(\beta)} \left(\frac{\alpha y}{2} \right)^{\beta-1} e^{-(\alpha y/2)/\alpha} \left| \frac{\alpha}{2} \right| \\
&= \frac{y^{\beta-1}}{2^\beta \Gamma(\beta)} e^{-y/2} \\
&= \frac{y^{n/2-1}}{2^{n/2} \Gamma(n/2)} e^{-y/2} \qquad y \geq 0,
\end{aligned}$$

which is the probability density function of a chi-square random variable with n degrees of freedom.

APPL verification: The APPL statements

```
assume(alpha > 0);
assume(beta > 0);
X := [[x -> (1 / (alpha ^ beta * GAMMA(beta))) * x ^ (beta - 1) * exp(-x / alpha)],
      [0, infinity], ["Continuous", "PDF"]];
g := [[x -> 2 * x / alpha], [0, infinity]];
Y := Transform(X, g);
subs(beta = n / 2, Y[1][1](x));
Z := ChiSquareRV(n);
```

yield identical functional forms

$$f(y) = \frac{y^{n/2-1}}{2^{n/2} \Gamma(n/2)} e^{-y/2} \qquad y > 0.$$