Theorem If $X_i \sim \text{gamma}(\alpha, \beta_i)$, for $i = 1, 2, \ldots, n$ and $X_1, X_2, \ldots, X_n$ are mutually independent random variables, then

$$\sum_{i=1}^{n} X_i \sim \text{gamma}\left(\alpha, \sum_{i=1}^{n} \beta_i\right).$$

Proof The moment generating function of X_i is

$$M_{X_i}(t) = (1 - \alpha t)^{-\beta_i} \qquad t < \frac{1}{\alpha}$$

for $i = 1,\ 2,\ \ldots,\ n$. Let $Y = X_1 + X_2 + \cdots + X_n$. Since the X_i's are mutually independent random variables, the moment generating function of Y is

$$\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E[e^{t(X_1+X_2+\cdots+X_n)}] \\
&= E(e^{tX_1}e^{tX_2}\ldots e^{tX_n}) \\
&= E[e^{tX_1}]E[e^{tX_2}]\ldots E[e^{tX_n}] \\
&= (1-\alpha t)^{-\beta_1}(1-\alpha t)^{-\beta_2}\ldots(1-\alpha t)^{-\beta_n} \\
&= (1-\alpha t)^{-\sum_{i=1}^{n}\beta_i} \qquad t < 1/\alpha,
\end{aligned}$$

which is the moment generating function of a gamma $(\alpha, \sum_{i=1}^{n} \beta_i)$ random variable.

APPL illustration: The APPL statements

```
assume(alpha > 0);
assume(beta1 > 0);
assume(beta2 > 0);
X1 := [[x -> (1 / (alpha ^ beta1 * GAMMA(beta1))) * x ^ (beta1 - 1) *
            exp(-x / alpha)], [0, infinity], ["Continuous", "PDF"]];
X2 := [[x -> (1 / (alpha ^ beta2 * GAMMA(beta2))) * x ^ (beta2 - 1) *
            exp(-x / alpha)], [0, infinity], ["Continuous", "PDF"]];
simplify(MGF(X1));
simplify(MGF(X2));
simplify(MGF(X1) * MGF(X2));
```

yield the moment generating function

$$M_{X_1+X_2}(t) = (1 - \alpha t)^{-(\beta_1+\beta_2)} \qquad t < \frac{1}{\alpha}.$$

The result holds for $n > 2$ by induction.