Theorem If X_1 and X_2 are independent random variables and $X_1 \sim$gamma(α, β_1) and $X_2 \sim$gamma(α, β_2), then the random variable $\frac{X_1}{X_1+X_2}$ has the beta distribution.

Proof Let X_1 and X_2 have the gamma distribution with probability density function

$$f(x) = \frac{1}{\alpha^{\beta_i}\Gamma(\beta_i)} x^{\beta_i - 1} e^{-x/\alpha} \qquad x > 0,$$

for $i = 1, 2$. Consider the transformation $Y_1 = \frac{X_1}{X_1+X_2}$ and the dummy transformation $Y_2 = X_1 + X_2$. This is a 1–1 transformation. So the transformation is

$$\begin{aligned} Y_1 &= g_1(X_1, X_2) = \frac{X_1}{X_1 + X_2} \\ Y_2 &= g_2(X_1, X_2) = X_1 + X_2. \end{aligned}$$

Solving for X_1 and X_2,

$$\begin{aligned} X_1 &= g_1^{-1}(Y_1, Y_2) = Y_1 Y_2 \\ X_2 &= g_1^{-1}(Y_1, Y_2) = Y_2(1 - Y_1) \end{aligned}$$

and Jacobian

$$J = \begin{vmatrix} Y_2 & Y_1 \\ -Y_2 & 1 - Y_1 \end{vmatrix} = Y_2(1 - Y_1) + Y_1 Y_2 = Y_2.$$

Because X_1 and X_2 are independent random variables, the joint probability density function of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} x_1^{\beta_1-1} x_2^{\beta_2-1} e^{-(x_1+x_2)/\alpha} \qquad x_1 > 0, x_2 > 0.$$

Therefore by the transformation technique, the probability density function of Y_1 and Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2))\,|J| \\ &= \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} (y_1 y_2)^{\beta_1-1} (y_2(1 - y_1))^{\beta_2-1} e^{-y_2/\alpha} |y_2| \end{aligned}$$

for $0 < y_1 < 1, y_2 > 0$. To calculate the marginal distribution $f_{Y_1}(y_1)$, integrate with respect to y_2:

$$\begin{aligned} f_{Y_1}(y_1) &= \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty (y_1 y_2)^{\beta_1-1} (y_2(1 - y_1))^{\beta_2-1} e^{-y_2/\alpha} |y_2|\, dy_2 \\ &= \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty y_1^{\beta_1-1} y_2^{\beta_1-1} y_2^{\beta_2-1} e^{-y_2/\alpha} y_2 (1 - y_1)^{\beta_2-1} dy_2 \\ &= \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty y_2^{\beta_1+\beta_2-1} e^{-y_2/\alpha} y_1^{\beta_1-1} (1 - y_1)^{\beta_2-1} dy_2 \qquad 0 < y_1 < 1. \end{aligned}$$

Let $r = y_2/\alpha$ and $dy_2 = \alpha\, dr$. The marginal distribution becomes

$$f_{Y_1}(y_1) = \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty (r\alpha)^{\beta_1+\beta_2-1} e^{-r} y_1^{\beta_1-1} (1 - y_1)^{\beta_2-1} \alpha\, dr \qquad 0 < y_1 < 1.$$

Using the definition of the gamma function for $\Gamma(\beta_1 + \beta_2)$, this becomes

$$f_{Y_1}(y_1) = \frac{\Gamma(\beta_1 + \beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} y_1^{\beta_1 - 1}(1 - y_1)^{\beta_2 - 1} \qquad 0 < y_1 < 1$$

which is the probability density function of a beta random variable.

APPL verification: The bivariate transformation function must be used to verify the derivation.