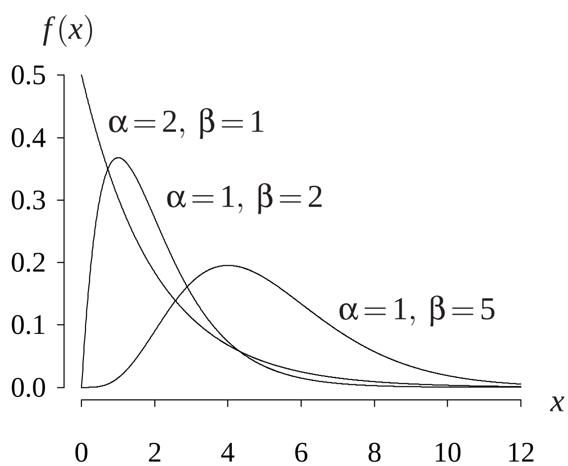
Gamma distribution (from http://www.math.wm.edu/~leemis/chart/UDR/UDR.html)
The shorthand $X \sim \text{gamma}(\alpha, \beta)$ is used to indicate that the random variable X has the gamma distribution. A gamma random variable X with positive scale parameter α and positive shape parameter β has probability density function

$$f(x) = \frac{x^{\beta-1} e^{-x/\alpha}}{\alpha^{\beta} \Gamma(\beta)} \qquad x > 0.$$

The gamma distribution can be used to model service times, lifetimes of objects, and repair times. The gamma distribution has an exponential right-hand tail. The probability density function with several parameter combinations is illustrated below.



The cumulative distribution function on the support of X is

$$F(x) = P(X \le x) = \frac{\Gamma(\beta, x/\alpha)}{\Gamma(\beta)} \qquad x > 0,$$

where

$$\Gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$$

for $s > 0$ and $x > 0$ is the incomplete gamma function and

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

for $s > 0$ is the gamma function. The survivor function on the support of X is

$$S(x) = P(X \ge x) = 1 - \frac{\Gamma(\beta, x/\alpha)}{\Gamma(\beta)} \qquad x > 0.$$

The hazard function on the support of X is

$$h(x) = \frac{f(x)}{S(x)} = \frac{x^{\beta-1} e^{-x/\alpha}}{\left(\Gamma(\beta) - \Gamma(\beta, x/\alpha)\right) \alpha^{\beta} \Gamma(\beta)} \qquad x > 0.$$

The cumulative hazard function on the support of X is

$$H(x) = -\ln S(x) = -\ln\left(1 - \frac{\Gamma(\beta, x/\alpha)}{\Gamma(\beta)}\right) \qquad x > 0.$$

There is no closed-form expression for the inverse distribution function. The moment generating function of X is

$$M(t) = E\left[e^{tX}\right] = (1 - \alpha t)^{-\beta} \qquad t < \frac{1}{\alpha}.$$

The characteristic function of X is

$$\phi(t) = E\left[e^{itX}\right] = (1 - \alpha i t)^{-\beta} \qquad t < \frac{1}{\alpha}.$$

The population mean, variance, skewness, and kurtosis of X are

$$E[X] = \alpha\beta \qquad V[X] = \alpha^2\beta \qquad E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{2}{\sqrt{\beta}} \qquad E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = 3 + \frac{6}{\beta}.$$

For $X_1, X_2, \ldots, X_n$ mutually independent gamma(α, β) random variables, the method of moments for α and β are

$$\hat{\alpha} = s^2/\bar{x}$$

and

$$\hat{\beta} = (\bar{x}/s)^2.$$

APPL verification: The APPL statements

```
assume(alpha > 0);
assume(beta > 0);
X := [[x -> x ^ (beta - 1) * exp(-x / alpha) / (alpha ^ beta * GAMMA(beta))],
      [0, infinity], ["Continuous", "PDF"]]:
Mean(X);
Variance(X);
Skewness(X);
Kurtosis(X);
```

verify the population mean, variance, skewness, and kurtosis.