Theorem The reciprocal of an $F(n_1, n_2)$ random variable is an $F(n_2, n_1)$ random variable.

Proof Let the random variable X have the F distribution with probability density function

$$f_X(x) = \frac{\Gamma((n_1+n_2)/2)(n_1/n_2)^{n_1/2} x^{n_1/2-1}}{\Gamma(n_1/2)\Gamma(n_2/2)[(n_1/n_2)x+1]^{(n_1+n_2)/2}} \qquad x > 0.$$

The transformation $Y = g(X) = 1/X$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, x > 0\}$ to $\mathcal{Y} = \{y \,|\, y > 0\}$ with inverse $X = g^{-1}(Y) = 1/Y$ and Jacobian

$$\frac{dX}{dY} = -Y^{-2}.$$

By the transformation technique, the probability density function of Y is

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right| \\
&= \frac{\Gamma((n_1+n_2)/2)}{\Gamma(n_1/2)\Gamma(n_2/2)} \frac{(n_1/n_2)^{n_1/2}(1/y)^{n_1/2-1}}{[(n_1/n_2)(1/y)+1]^{(n_1+n_2)/2}} |-y^{-2}| \\
&= \frac{\Gamma((n_2+n_1)/2)}{\Gamma(n_2/2)\Gamma(n_1/2)} \frac{n_1^{n_1/2}}{n_2^{n_1/2}} \frac{(1/y)^{n_1/2-1}}{[\frac{n_1+n_2y}{n_2y}]^{(n_1+n_2)/2}} |-y^{-2}| \\
&= \frac{\Gamma((n_2+n_1)/2)}{\Gamma(n_2/2)\Gamma(n_1/2)} \left(\frac{n_1^{n_1/2}}{n_2^{n_1/2}}\right) \left(y^{1-n_1/2}\right) \left(\frac{n_2^{n_1/2} n_2^{n_2/2} y^{(n_1+n_2)/2}}{[n_1+n_2y]^{(n_1+n_2)/2}}\right) \left(y^{-2}\right) \\
&= \frac{\Gamma((n_2+n_1)/2)}{\Gamma(n_2/2)\Gamma(n_1/2)} \left(\frac{n_1^{n_1/2}}{n_2^{n_1/2}}\right) \left(y^{(1-n_1/2)+((n_1+n_2)/2)-2}\right) \left(\frac{n_2^{n_1/2} n_2^{n_2/2}}{[1+\frac{n_2}{n_1}y]^{((n_1+n_2)/2)} n_1^{n_1/2} n_1^{n_2/2}}\right) \\
&= \frac{\Gamma((n_2+n_1)/2)}{\Gamma(n_2/2)\Gamma(n_1/2)} \left(\frac{n_2^{n_2/2}}{n_1^{n_2/2}}\right) \left(\frac{y^{n_2/2-1}}{[1+\frac{n_2}{n_1}y]^{((n_1+n_2)/2)}}\right) \\
&= \frac{\Gamma((n_2+n_1)/2)}{\Gamma(n_2/2)\Gamma(n_1/2)} \left((n_2/n_1)^{n_2/2} \frac{y^{n_2/2-1}}{[1+\frac{n_2}{n_1}y]^{((n_1+n_2)/2)}}\right) \qquad y > 0,
\end{aligned}$$

which is the probability density function of the F distribution.

APPL verification: The APPL statements

```
X := FRV(n1, n2);
g := [[x -> 1 / x], [0, infinity]];
Y := Transform(X, g);
simplify(op(Y[1]));
FRV(n2, n1);
```

yield the same probability density functions after some algebraic manipulation.