Theorem If X is an exponential random variable, then $X^{1/\beta}$ is a Weibull random variable.

Proof Let the random variable X have the exponential distribution with probability density function

$$f_X(x) = \frac{1}{\alpha} e^{-x/\alpha} \qquad\qquad x > 0.$$

The transformation $Y = g(X) = X^{1/\beta}$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, x > 0\}$ to $\mathcal{Y} = \{y \,|\, y > 0\}$ with inverse $X = g^{-1}(Y) = Y^{\beta}$ and Jacobian

$$\frac{dX}{dY} = \beta Y^{\beta - 1}.$$

By the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\ &= \frac{1}{\alpha} \beta y^{\beta-1} e^{-y^{\beta}/\alpha} \\ &= \frac{\beta}{\alpha} y^{\beta-1} e^{-y^{\beta}/\alpha} \qquad\qquad y > 0, \end{aligned}$$

which is the probability density function of a Weibull random variable.

APPL verification: The APPL statements

```
assume(alpha > 0);
assume(beta > 0);
X := [[x -> exp(-x / alpha) / alpha], [0, infinity], ["Continuous", "PDF"]];
g := [[x -> x ^ (1 / beta)], [0, infinity]];
Y := Transform(X, g);
```

yield the probability density function given above.