Theorem The square root of an exponential random variable has the Rayleigh distribution.

Proof Let the random variable X have the exponential distribution with probability density function

$$f_X(x) = \frac{1}{\alpha} e^{-x/\alpha} \qquad\qquad x > 0.$$

The transformation $Y = g(X) = \sqrt{X}$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, x > 0\}$ to $\mathcal{Y} = \{y \,|\, y > 0\}$ with inverse $X = g^{-1}(Y) = Y^2$ and Jacobian

$$\frac{dX}{dY} = 2Y.$$

Therefore by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\ &= \frac{1}{\alpha} e^{-y^2/\alpha} \, |2y| \\ &= \frac{2y}{\alpha} e^{-y^2/\alpha} \qquad\qquad y > 0, \end{aligned}$$

which is the probability density function of the Rayleigh(α) distribution.

APPL verification: The APPL statements

```
assume(alpha > 0);
X := ExponentialRV(1 / alpha);
g := [[x -> sqrt(x)], [0, infinity]];
Y := Transform(X, g);
```

yield the probability density function of an Rayleigh(α) random variable

$$f_Y(y) = \frac{2y}{\alpha} e^{-y^2/\alpha} \qquad\qquad y > 0.$$