Theorem The distribution of the difference of two independent exponential random variables, with population means α_1 and α_2 respectively, has a Laplace distribution with parameters α_1 and α_2.

Proof Let X_1 and X_2 be independent exponential random variables with population means α_1 and α_2 respectively. Define $Y = X_1 - X_2$. The goal is to find the distribution of Y by the cumulative distribution function technique. First, we know that

$$f_{X_1}(x_1) = \frac{1}{\alpha_1}\, e^{-x_1/\alpha_1} \qquad\qquad x_1 > 0$$

and

$$f_{X_2}(x_2) = \frac{1}{\alpha_2}\, e^{-x_2/\alpha_2} \qquad\qquad x_2 > 0.$$

By independence, it follows that the joint probability density function of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\alpha_1\alpha_2}\, e^{-x_1/\alpha_1}\, e^{-x_2/\alpha_2} \qquad\qquad x_1 > 0, x_2 > 0.$$

We want $F_Y(y) = P(Y \le y)$ where $-\infty < y < \infty$. Looking at the support of $f_{X_1,X_2}(x_1, x_2)$ given above, we see that the resulting cumulative distribution function for Y will be piecewise, where the pieces are separated at $y = 0$. So we first consider the case where $y \le 0$. The double integral giving us the cumulative distribution function $F_Y(y)$ of Y for $y \le 0$ is

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(X_1 - X_2 \le y) \\
&= P(X_2 \ge X_1 - y) \\
&= \int_0^\infty \int_{x_1-y}^\infty f_{X_1,X_2}(x_1, x_2)\, dx_2 dx_1 \\
&= \int_0^\infty \int_{x_1-y}^\infty \frac{1}{\alpha_1\alpha_2}\, e^{-x_1/\alpha_1}\, e^{-x_2/\alpha_2}\, dx_2 dx_1 \\
&= \frac{\alpha_2}{\alpha_1+\alpha_2} e^{y/\alpha_2} \qquad\qquad y \le 0.
\end{aligned}$$

For $y > 0$, the cumulative distribution function of Y is

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= 1 - P(Y > y) \\
&= 1 - P(X_1 - X_2 > y) \\
&= 1 - P(X_2 \ge X_1 - y) \\
&= 1 - \int_y^\infty \int_0^{x_1-y} f_{X_1,X_2}(x_1, x_2)\, dx_2 dx_1 \\
&= 1 - \int_y^\infty \int_0^{x_1-y} \frac{1}{\alpha_1\alpha_2}\, e^{-x_1/\alpha_1}\, e^{-x_2/\alpha_2}\, dx_2 dx_1 \\
&= 1 - \frac{\alpha_1}{\alpha_1+\alpha_2} e^{-y/\alpha_1} \qquad\qquad y > 0.
\end{aligned}$$

Differentiating with respect to y gives

$$f_Y(y) = \begin{cases} \frac{e^{y/\alpha_2}}{\alpha_1+\alpha_2} & y \le 0, \\ \frac{e^{-y/\alpha_1}}{\alpha_1+\alpha_2} & y > 0, \end{cases}$$

which is the probability density function of a Laplace random variable with parameters α_1 and α_2.

APPL verification: The APPL `LaPlaceRV` function operates on the assumption that the independent exponential parameters are equal, i.e. that $\alpha_1 = \alpha_2$. So, APPL was not able to verify the general result. However, the APPL statements

```
assume(alpha1 > 0);
assume(alpha2 > 0);
X1 := [[x -> exp(-x / alpha1) / alpha1], [0, infinity], ["Continuous", "PDF"]];
X2 := [[x -> exp(-x / alpha2) / alpha2], [0, infinity], ["Continuous", "PDF"]];
Difference(X1, X2);
```

verify the probability density function of $Y = X_1 - X_2$ found above.