Theorem If X_1 and X_2 are independent and identically distributed exponential(1) random variables, then X_1/X_2 has the F distribution.

Proof Let X_1 and X_2 be independent exponential(1) random variables. We can write their probability density functions as

$$f_{X_1}(x_1) = e^{-x_1} \qquad\qquad x_1 > 0$$

and

$$f_{X_2}(x_2) = e^{-x_2} \qquad\qquad x_2 > 0.$$

Since X_1 and X_2 are independent, the joint probability density function of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = e^{-(x_1+x_2)} \qquad\qquad x_1 > 0, x_2 > 0.$$

Consider the 2×2 transformation

$$Y_1 = g_1(X_1, X_2) = \frac{X_1}{X_2} \qquad \text{and} \qquad Y_2 = g_2(X_1, X_2) = X_2$$

which is a 1–1 transformation from $\mathcal{X} = \{(x_1, x_2) \,|\, x_1 > 0, x_2 > 0\}$ to $\mathcal{Y} = \{(y_1, y_2) \,|\, y_1 > 0, y_2 > 0\}$ with inverses

$$X_1 = g_1^{-1}(Y_1, Y_2) = Y_1 Y_2 \qquad \text{and} \qquad X_2 = g_2^{-1}(Y_1, Y_2) = Y_2$$

and Jacobian

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = Y_2.$$

Therefore, by the transformation technique, the joint probability density function of Y_1 and Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}\left(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\right)|J| \\ &= e^{-(y_1 y_2 + y_2)}\,|y_2| \\ &= y_2 e^{-(y_1 y_2 + y_2)} \qquad\qquad y_1 > 0, y_2 > 0. \end{aligned}$$

for $y_1 > 0, y_2 > 0$. Using integration by parts, the probability density function of Y_1 is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^\infty f_{Y_1,Y_2}(y_1, y_2)\, dy_2 \\ &= \int_0^\infty y_2 e^{-(y_1 y_2 + y_2)} dy_2 \\ &= \frac{1}{(1+y_1)^2} \qquad\qquad y_1 > 0, \end{aligned}$$

which is the probability density function of a F random variable with $n_1 = 2$ and $n_2 = 2$ degrees of freedom.

APPL verification: The APPL statements

```
X1 := ExponentialRV(1);
X2 := ExponentialRV(1);
g := [[x -> 1 / x], [0, infinity]];
Y := Transform(X2, g);
Product(X1, Y);
FRV(2, 2);
```

produce the probability density function of a F random variable with $n_1 = 2$ and $n_2 = 2$ degrees of freedom.