Theorem The sum of n mutually independent exponential random variables, each with common population mean $\alpha > 0$ is an Erlang(α, n) random variable.

Proof Let $X_1, X_2, \ldots, X_n$ be mutually independent exponential random variables with common population mean $\alpha > 0$, each have probability density function

$$f_{X_i}(x) = \frac{1}{\alpha} e^{-x/\alpha} \qquad\qquad x > 0,$$

for $i = 1, 2, \ldots, n$. The moment generation function of X_i is

$$M_{X_i}(t) = (1 - \alpha t)^{-1} \qquad\qquad t < \frac{1}{\alpha},$$

for $i = 1, 2, \ldots, n$. Since the random variables $X_1, X_2, \ldots, X_n$ are mutually independent, the moment generation function of $X = \sum_{i=1}^{n} X_i$ is

$$\begin{aligned}
M_X(t) &= E\left[e^{tX}\right] \\
&= E\left[e^{t\sum_{i=1}^{n} X_i}\right] \\
&= E\left[e^{tX_1} e^{tX_2} \ldots e^{tX_n}\right] \\
&= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \ldots E\left[e^{tX_n}\right] \\
&= M_{X_1}(t) M_{X_2}(t) \ldots M_{X_n}(t) \\
&= (1 - \alpha t)^{-1} (1 - \alpha t)^{-1} \ldots (1 - \alpha t)^{-1} \\
&= (1 - \alpha t)^{-n} \qquad\qquad t < \frac{1}{\alpha},
\end{aligned}$$

which is the moment generation function of an Erlang(α, n) random variable.

APPL verification: The following APPL statements verify a special case ($n = 3$) of this result.

```
assume(alpha > 0);
X := [[x -> exp(-x / alpha) / alpha], [0, infinity],
      ["Continuous", "PDF"]];
Y := ConvolutionIID(X, 3);
```

In this case Y has an Erlang$(\alpha, 3)$ distribution.