Theorem If $X_1, X_2, \ldots, X_n$ are mutually independent exponential random variables each with mean $\alpha > 0$, then $X = \frac{2}{\alpha}\sum_{i=1}^{n} X_i$ is a $\chi^2(2n)$ random variable.

Proof Since the random variables $X_1, X_2, \ldots, X_n$ have the exponential distribution with mean α, they each have probability density function

$$f_{X_i}(x) = \frac{1}{\alpha}\, e^{-x/\alpha} \qquad\qquad x > 0,$$

for $i = 1,\, 2,\, \ldots,\, n$. The moment generation function of X_i is

$$M_{X_i}(t) = (1 - \alpha t)^{-1} \qquad\qquad t < \frac{1}{\alpha},$$

for $i = 1,\, 2,\, \ldots,\, n$. Since the random variables $X_1, X_2, \ldots, X_n$ are mutually independent, the moment generation function of $X = \frac{2}{\alpha}\sum_{i=1}^{n} X_i$ is

$$\begin{aligned}
M_X(t) &= E\left[e^{tX}\right] \\
&= E\left[e^{(2t/\alpha)\sum_{i=1}^{n} X_i}\right] \\
&= E\left[e^{t\sum_{i=1}^{n} 2X_i/\alpha}\right] \\
&= E\left[e^{t\cdot 2X_1/\alpha} e^{t\cdot 2X_2/\alpha} \ldots e^{t\cdot 2X_n/\alpha}\right] \\
&= E\left[e^{t\cdot 2X_1/\alpha}\right] E\left[e^{t\cdot 2X_2/\alpha}\right] \ldots E\left[e^{t\cdot 2X_n/\alpha}\right] \\
&= M_{X_1}(2t/\alpha) M_{X_2}(2t/\alpha) \ldots M_{X_n}(2t/\alpha) \\
&= (1-2t)^{-1}(1-2t)^{-1} \ldots (1-2t)^{-1} \\
&= (1-2t)^{-n} \qquad\qquad t < 1/2,
\end{aligned}$$

which is the moment generation function of a $\chi^2(2n)$ random variable.

APPL verification: The following APPL statements verify a special case ($n = 3$) of this result.

```
assume(alpha > 0);
X1 := [[x -> exp(-x / alpha) / alpha], [0, infinity], ["Continuous", "PDF"]];
X2 := [[x -> exp(-x / alpha) / alpha], [0, infinity], ["Continuous", "PDF"]];
X3 := [[x -> exp(-x / alpha) / alpha], [0, infinity], ["Continuous", "PDF"]];
Y  := Convolution(X1, X2, X3);
g  := [[x -> 2 * x / alpha], [0, infinity]];
X  := Transform(Y, g);
```

In this case X has a $\chi^2(6)$ distribution.