Theorem The Erlang distribution has the scaling property. That is, if $X \sim \text{Erlang}(\alpha, n)$ then $Y = kX$ also has the Erlang distribution.

Proof Let the random variable X have the Erlang(α, n) distribution with probability density function

$$f(x) = \frac{x^{n-1}e^{-x/\alpha}}{\alpha^n(n-1)!} \qquad\qquad x > 0.$$

Let k be a positive, real constant. The transformation $Y = g(X) = kX$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, x > 0\}$ to $\mathcal{Y} = \{y \,|\, y > 0\}$ with inverse $X = g^{-1}(Y) = Y/k$ and Jacobian

$$\frac{dX}{dY} = \frac{1}{k}.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| \\ &= \frac{(y/k)^{n-1}e^{-y/k\alpha}}{\alpha^n(n-1)!}\left|\frac{1}{k}\right| \\ &= \frac{y^{n-1}e^{-y/k\alpha}}{(k\alpha)^n(n-1)!} \qquad\qquad y > 0, \end{aligned}$$

which is the probability density function of a Erlang$(k\alpha, n)$ random variable.

APPL verification: The APPL statements

```
assume(k > 0);
X := [[x -> x ^ (n - 1) * exp(-x / alpha) / (alpha ^ n * factorial(n - 1))],
      [0, infinity], ["Continuous", "PDF"]];
g := [[x -> k * x], [0, infinity]];
Y := Transform(X, g);
```

yield the probability density function of a Erlang$(k\alpha, n)$ random variable.