**Theorem** If $X_1 \sim \chi^2(n_1)$ and $X_2 \sim \chi^2(n_2)$ are independent random variables, then

$$\frac{X_1/n_1}{X_2/n_2} \sim F(n_1, n_2).$$

**Proof** The random variable $X_1 \sim \chi^2(n_1)$ has probability density function

$$f_{X_1}(x_1) = \frac{1}{2^{n_1/2}\Gamma(n_1/2)} x_1^{n_1/2-1} e^{-x_1/2} \qquad x_1 > 0.$$

Likewise, the random variable $X_2 \sim \chi^2(n_2)$ has probability density function

$$f_{X_2}(x_2) = \frac{1}{2^{n_2/2}\Gamma(n_2/2)} x_2^{n_2/2-1} e^{-x_2/2} \qquad x_2 > 0.$$

Since $X_1$ and $X_2$ are independent, the joint probability density function of $X_1$ and $X_2$ is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2^{n_1/2}\Gamma(n_1/2)} x_1^{n_1/2-1} e^{-x_1/2} \frac{1}{2^{n_2/2}\Gamma(n_2/2)} x_2^{n_2/2-1} e^{-x_2/2} \qquad x_1 > 0, x_2 > 0.$$

Consider the $2 \times 2$ transformation

$$Y_1 = g_1(X_1, X_2) = \frac{X_1/n_1}{X_2/n_2} \qquad \text{and} \qquad Y_2 = g_2(X_1, X_2) = X_2$$

which is a 1–1 transformation from $\mathcal{X} = \{(x_1, x_2) \,|\, x_1 > 0, x_2 > 0\}$ to $\mathcal{Y} = \{(y_1, y_2) \,|\, y_1 > 0, y_2 > 0\}$ with inverses

$$X_1 = g_1^{-1}(Y_1, Y_2) = \frac{n_1 Y_1 Y_2}{n_2} \qquad \text{and} \qquad X_2 = g_2^{-1}(Y_1, Y_2) = Y_2$$

and Jacobian

$$J = \begin{vmatrix} \frac{n_1 Y_2}{n_2} & \frac{n_1 Y_1}{n_2} \\ 0 & 1 \end{vmatrix} = \frac{n_1 Y_2}{n_2}.$$

Therefore by the transformation technique, the joint probability density function of $Y_1$ and $Y_2$ is

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}\left(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\right)|J| \\ &= \frac{1}{2^{n_1/2}\Gamma(n_1/2)} \left(\frac{n_1 y_1 y_2}{n_2}\right)^{n_1/2-1} e^{-n_1 y_1 y_2/(2n_2)} \frac{1}{2^{n_2/2}\Gamma(n_2/2)} y_2^{n_2/2-1} e^{-y_2/2} \left|\frac{n_1 y_2}{n_2}\right| \end{aligned}$$

for $y_1 > 0, y_2 > 0$. In order to find the probability density function of $Y_1$,

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^\infty f_{Y_1,Y_2}(y_1, y_2)\, dy_2 \\ &= \int_0^\infty \frac{1}{2^{n_1/2}\Gamma(n_1/2)} \left(\frac{n_1 y_1 y_2}{n_2}\right)^{n_1/2-1} e^{-n_1 y_1 y_2/(2n_2)} \frac{1}{2^{n_2/2}\Gamma(n_2/2)} y_2^{n_2/2-1} e^{-y_2/2} \left(\frac{n_1 y_2}{n_2}\right) dy_2 \\ &= \frac{\Gamma((n_1+n_2)/2)(n_1/n_2)^{n_1/2} y_1^{n_1/2-1}}{\Gamma(n_1/2)\Gamma(n_2/2)[(n_1/n_2)y_1 + 1]^{((n_1+n_2)/2)}} \qquad y_1 > 0 \end{aligned}$$

by using a change-of-variable to perform the integration. This is the probability density function of an $F$ random variable with $n_1$ and $n_2$ degrees of freedom.

**APPL demonstration:** The APPL statements

```
X1 := ChiSquareRV(k1);
X2 := ChiSquareRV(k2);
g1 := [[x -> x / k1], [0, infinity]];
g2 := [[x -> k2 / x], [0, infinity]];
Y1 := Transform(X1, g1);
Y2 := Transform(X2, g2);
F  := Product(Y1, Y2);
```

return the probability density function of an $F$ random variable with the appropriate degrees of freedom.