Chi-square distribution (from [http://www.math.wm.edu/~leemis/chart/UDR/UDR.html](http://www.math.wm.edu/~leemis/chart/UDR/UDR.html))

The shorthand $X \sim \chi^2(n)$ is used to indicate that the random variable $X$ has the chi-square distribution with positive real-valued parameter $n$, which is known as the degrees of freedom. In most applications, $n$ is a positive integer. A chi-square random variable $X$ with $n$ degrees of freedom has probability density function

$$f(x) = \frac{x^{n/2-1}e^{-x/2}}{2^{n/2}\Gamma(n/2)}, \quad x > 0,$$

for $n = 1, 2, \ldots$. The chi-square distribution is used for inference concerning observations drawn from an exponential population and in determining the critical values for the chi-square goodness-of-fit test. The probability density function with $n = 1, 2, 3$ is illustrated below.

The cumulative distribution function on the support of $X$ is

$$F(x) = P(X \leq x) = \frac{\Gamma(n/2, x/2)}{\Gamma(n/2)} \quad x > 0.$$

The survivor function on the support of $X$ is

$$S(x) = P(X \geq x) = 1 - \frac{\Gamma(n/2, x/2)}{\Gamma(n/2)} \quad x > 0.$$

The hazard function $h(x) = f(x)/S(x)$ and the cumulative hazard function $H(x) = -\ln S(x)$ can be written in terms of the gamma and incomplete gamma functions. The inverse distribution function of $X$ can't be expressed in closed form (except when $n = 2$). The mode of $X$ is

$$\max\{n-2, 0\} \quad n > 2.$$

The moment generating function of $X$ is

$$M(t) = E[e^{tX}] = (1 - 2t)^{-n/2} \quad t < 1/2.$$
The characteristic function of $X$ is
\[
\phi(t) = E \left[ e^{itX} \right] = (1 - 2it)^{-n/2} \quad t < 1/2.
\]

The population mean, variance, skewness, and kurtosis of $X$ are
\[
E[X] = n \quad V[X] = 2n \quad E \left[ \left( \frac{X - \mu}{\sigma} \right)^3 \right] = \sqrt{8/n} \quad E \left[ \left( \frac{X - \mu}{\sigma} \right)^4 \right] = 3 + \frac{12}{n}.
\]

**APPL verification:** The APPL statements

\[
X := \text{ChiSquareRV}(n);
\text{CDF}(X);
\text{SF}(X);
\text{Mean}(X);
\text{Variance}(X);
\text{Skewness}(X);
\text{Kurtosis}(X);
\text{MGF}(X);
\]

verify the cumulative distribution function, survivor function, population mean, variance, skewness, kurtosis, and moment generating function.