**Theorem** [UNDER CONSTRUCTION!] The Cauchy distribution has the convolution property. That is, if $X_i \sim \text{Cauchy}(a_i, \alpha_i)$, $i = 1, 2, \ldots, n$, are independent random variables then $Y = \sum_{i=1}^{n} X_i$ also has the Cauchy distribution.

**Proof** [UNDER CONSTRUCTION!] Let the random variable $X_1$ have the Cauchy$(a_1, \alpha_1)$ distribution with probability density function

$$f_{X_1}(x_1) = \frac{1}{\alpha_1 \pi [1 + ((x_1 - a_1)/\alpha_1)^2]} \qquad -\infty < x_1 < \infty.$$

Let the random variable $X_2$ have the Cauchy$(a_2, \alpha_2)$ distribution with probability density function

$$f_{X_2}(x_2) = \frac{1}{\alpha_2 \pi [1 + ((x_2 - a_2)/\alpha_2)^2]} \qquad -\infty < x_2 < \infty.$$

Assume $X_1$ and $X_2$ are independent. The joint probability density function of $X_1$ and $X_2$ is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\alpha_1 \alpha_2 \pi^2 [1 + ((x_1 - a_1)/\alpha_1)^2][1 + ((x_2 - a_2)/\alpha_2)^2]}$$

for $-\infty < x_1 < \infty, -\infty < x_2 < \infty$. Consider the $2 \times 2$ transformation

$$Y_1 = g_1(X_1, X_2) = X_1 + X_2 \qquad \text{and} \qquad Y_2 = g_2(X_1, X_2) = X_2$$

which is a 1–1 transformation from $\mathcal{X} = \{(x_1, x_2) \mid -\infty < x_1 < \infty, -\infty < x_2 < \infty\}$ to $\mathcal{Y} = \{(y_1, y_2) \mid -\infty < y_1 < \infty, -\infty < y_2 < \infty\}$ with inverses

$$X_1 = g_1^{-1}(Y_1, Y_2) = Y_1 - Y_2 \qquad \text{and} \qquad X_2 = g_2^{-1}(Y_1, Y_2) = Y_2$$

and Jacobian

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore, by the transformation technique, the joint probability density function of $Y_1$ and $Y_2$ is

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}\left(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)\right) |J| \\ &= \frac{1}{\alpha_1 \alpha_2 \pi^2 [1 + ((y_1 - y_2 - a_1)/\alpha_1)^2][1 + ((y_2 - a_2)/\alpha_2)^2]} \end{aligned}$$

for $-\infty < y_1 < \infty, -\infty < y_2 < \infty$. The probability density function of $Y_1$ is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2 \\ &= \int_{-\infty}^{\infty} \frac{1}{\alpha_1 \alpha_2 \pi^2 [1 + ((y_1 - y_2 - a_1)/\alpha_1)^2][1 + ((y_2 - a_2)/\alpha_2)^2]} dy_2 \end{aligned}$$

which is a difficult integral to evaluate. The Maple code

```
assume(alpha1 > 0);
assume(alpha2 > 0);
int(1 / ((1 + ((y1 - y2 -a1) / alpha1) ^ 2) * (1 + ((y2- a2) / alpha2) ^ 2)),
    y2 = -infinity .. infinity) / (alpha1 * alpha2 * Pi ^ 2);
```

gives an expression with complex variables. In special cases (for example, $a_1 = a_2 = 0$ and $\alpha_1 = \alpha_2 = 1$) the general result from Forbes, Evans, Hastings, and Peacock (Statistical Distributions, Fourth Edition, John Wiley and Sons, 2011, page 67)

$$Y \sim \text{Cauchy}\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} \alpha_i\right)$$

is confirmed.

**APPL failure:** The APPL statements

```
assume(alpha1 > 0);
assume(alpha2 > 0);
X := [[x -> 1 / (alpha1 * Pi * (1 + ((x - a1) / alpha1) ^ 2)),
       x -> 1 / (alpha1 * Pi * (1 + ((x - a1) / alpha1) ^ 2))],
      [-infinity, 0, infinity], ["Continuous", "PDF"]];
Y := [[y -> 1 / (alpha2 * Pi * (1 + ((y - a2) / alpha2) ^ 2)),
       y -> 1 / (alpha2 * Pi * (1 + ((y - a2) / alpha2) ^ 2))],
      [-infinity, 0, infinity], ["Continuous", "PDF"]];
Z := Convolution(X, Y);
```

fail to confirm the result for $n = 2$.