Theorem If $X \sim \text{binomial}(n, p)$ and $p \sim \text{beta}(a, b)$ then the probability density function of X is

$$f_X(x) = \frac{\Gamma(x+a)\Gamma(n-x+b)\Gamma(a+b)\Gamma(n+1)}{\Gamma(a+b+n)\Gamma(a)\Gamma(b)\Gamma(x+1)\Gamma(n-x+1)} \qquad x = 0, 1, 2, \ldots, n,$$

which is known as the beta–binomial distribution.

Proof The unconditional distribution of X (also known as the *compound distribution*) is

$$\begin{aligned}
f_X(x) &= \int_0^1 f_p(p) f_{X|p}(x \,|\, p)\, dp \\
&= \int_0^1 \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1}\right]\left[\binom{n}{x} p^x (1-p)^{n-x}\right] dp \\
&= \binom{n}{x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^{a+x-1}(1-p)^{n-x+b-1}\, dp \\
&= \binom{n}{x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+x)\Gamma(b+n-x)}{\Gamma(a+b+n)} \\
&= \frac{\Gamma(n+1)\Gamma(a+b)\Gamma(a+x)\Gamma(b+n-x)}{\Gamma(x+1)\Gamma(n-x+1)\Gamma(a)\Gamma(b)\Gamma(a+b+n)} \qquad x = 0, 1, 2, \ldots, n.
\end{aligned}$$

APPL verification: The APPL statements

```
Y := BetaRV(a, b);
X := BinomialRV(r, p);
int(Y[1][1](p) * X[1][1](x), p = 0 .. 1);
```

yield the probability density function of the beta–binomial distribution indicated in the theorem.