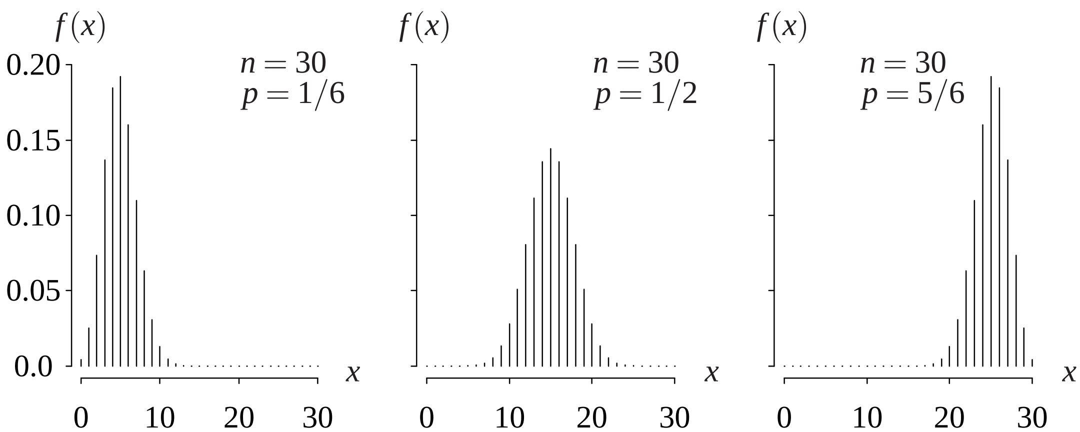
Binomial distribution (from http://www.math.wm.edu/~leemis/chart/UDR/UDR.html)
The shorthand $X \sim \text{binomial}(n, p)$ is used to indicate that the random variable X has the binomial distribution for positive integer parameter n and real parameter p satisfying $0 < p < 1$. The binomial distribution models the number of successes in n mutually independent Bernoulli trials, each with probability of success p. The random variable $X \sim \text{binomial}(n, p)$ has probability mass function

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n.$$

The binomial distribution can be used to model the number of people in a group of n people with a particular characteristic, the number of defective items in a batch of n items, the number of fours in n rolls of a fair die, or the number of rainy days in a month. Stated more generically, a binomial random variable is the number of successes in n mutually independent Bernoulli trials. Three illustrations of the shape of the probability mass function for $n = 30$ and $p = 1/6, 1/2, 5/6$ are given below.



The cumulative distribution function on the support of X is

$$F(x) = P(X \le x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} \qquad x = 0, 1, 2, \ldots, n.$$

The survivor function on the support of X is

$$S(x) = P(X \ge x) = \sum_{k=x}^{n} \binom{n}{k} p^k (1-p)^{n-k} \qquad x = 0, 1, 2, \ldots, n.$$

The moment generating function of X is

$$M(t) = E\left[e^{tX}\right] = \left(1 - p + pe^t\right)^n \qquad -\infty < t < \infty.$$

The characteristic function of X is

$$\phi(t) = E\left[e^{itX}\right] = \left(1 - p + pe^{it}\right)^n \qquad -\infty < t < \infty.$$

The population mean and variance of a binomial$(n,\, p)$ random variable are

$$E[X] = np \qquad\qquad V[X] = np(1-p)$$

and the population skewness and kurtosis are

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{1-2p}{\sqrt{np(1-p)}} \qquad\qquad E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = 3 + \frac{1-6p(1-p)}{np(1-p)}.$$

The population skewness and kurtosis converge to 0 and 3, respectively, in the limit as $n \to \infty$.

Let $x_1, x_2, \ldots, x_n$ be realizations of mutually independent Bernoulli(p) random variables. Assume that n is a fixed constant and that p is an unknown parameter satisfying $0 < p < 1$. The maximum likelihood estimator for p is

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

which is an unbiased estimator of p, that is $E[\hat{p}] = p$. An approximate $(1-\alpha)\cdot 100\%$ confidence interval for p is

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where $z_{\alpha/2}$ is the $1-\alpha/2$ percentile of the standard normal distribution. This confidence interval is symmetric about $\hat{p}$ and allows for an upper limit that is greater than 1 and a lower limit that is less than 0. A second approximate $(1-\alpha)\cdot 100\%$ confidence interval for p is

$$\frac{\hat{p} + \frac{z_{\alpha/2}^2}{2n} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{z_{\alpha/2}^2}{4n^2}}}{1 + z_{\alpha/2}^2/n} < p < \frac{\hat{p} + \frac{z_{\alpha/2}^2}{2n} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{z_{\alpha/2}^2}{4n^2}}}{1 + z_{\alpha/2}^2/n}.$$

A third approximate $(1-\alpha)\cdot 100\%$ confidence interval for p that is based on the Poisson approximation to the binomial distribution is

$$\frac{1}{2n}\chi^2_{2y,\,1-\alpha/2} < p < \frac{1}{2n}\chi^2_{2(y+1),\,\alpha/2},$$

where $y = x_1 + x_2 + \cdots + x_n$ and $\chi^2_{q,\beta}$ is the $1-\beta$ percentile of a chi-square distribution with q degrees of freedom. A fourth approximate $(1-\alpha)\cdot 100\%$ confidence interval for p is

$$\frac{1}{1 + \frac{n-y+1}{yF_{2y,\,2(n-y+1),\,1-\alpha/2}}} < p < \frac{1}{1 + \frac{n-y}{(y+1)F_{2(y+1),\,2(n-y),\,\alpha/2}}},$$

where $F_{q,r,\beta}$ is the $1-\beta$ percentile of an F random variable with q and r degrees of freedom.

APPL verification: The APPL statements

```
X := BinomialRV(n,p);
Mean(X);
Variance(X);
Skewness(X);
Kurtosis(X);
MGF(X);
```

verify the population mean, variance, skewness, kurtosis, and moment generating function.