Theorem The beta(b,b) distribution converges to the normal distribution when $b \to \infty$.

Proof (by Professor Robin Ryder in the CEREMADE at Université Paris Dauphine) Let the random variable X have the beta(b,b) distribution with probability density function

$$f_X(x) = \frac{\Gamma(2b)x^{b-1}(1-x)^{b-1}}{\Gamma(b)\Gamma(b)} \qquad 0 < x < 1,$$

where b is a real, positive parameter. The mean of X is $E[X] = 1/2$ and the variance of X is $V[X] = 1/4(2b+1)$. Substract the mean and divide by the standard deviation before taking the limit. So consider the transformation $Y = g(X) = 2\sqrt{2b+1}(X - 1/2)$, which is a one-to-one transformation from $\mathcal{A} = \{x | 0 < x < 1\}$ to $\mathcal{B} = \{y | -\sqrt{2b+1} < y < \sqrt{2b+1}\}$ with inverse $X = g^{-1}(Y) = X/2\sqrt{2b+1} + 1/2$ and Jacobian

$$\frac{dX}{dY} = \frac{1}{2\sqrt{2b+1}}.$$

Then the probability density function of Y is

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\sqrt{2b+1}} f_X\left(\frac{y}{2\sqrt{2b+1}} + \frac{1}{2}\right) \\
&= \frac{1}{2\sqrt{2b+1}} \cdot \frac{\Gamma(2b)}{\Gamma(b)^2} \left(\frac{1}{2} + \frac{y}{2\sqrt{2b+1}}\right)^{b-1} \left(\frac{1}{2} - \frac{y}{2\sqrt{2b+1}}\right)^{b-1} \\
&= \frac{1}{2\sqrt{2b+1}} \cdot \frac{\Gamma(2b)}{\Gamma(b)^2} \left(\frac{1}{4} - \frac{y^2}{4(2b+1)}\right)^{b-1} \qquad -\sqrt{2b+1} < y < \sqrt{2b+1}.
\end{aligned}$$

We now apply Stirling's formula $\Gamma(z) = \sqrt{2\pi/z}(z/e)^z(1 + O(1/z))$ and get

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\sqrt{2b+1}} \cdot \frac{\sqrt{\frac{2\pi}{2b}}\left(\frac{2b}{e}\right)^{2b}}{\frac{2\pi}{b}\left(\frac{b}{e}\right)^{2b}} \left(1 + O\left(\frac{1}{b}\right)\right) \left(\frac{1}{4} - \frac{y^2}{4(2b+1)}\right)^{b-1} \\
&= \frac{b}{\sqrt{2b}\sqrt{2b+1}} \cdot \frac{2^{2b}}{2\sqrt{2\pi}} \left(1 + O\left(\frac{1}{b}\right)\right) \left(\frac{1}{4} - \frac{y^2}{4(2b+1)}\right)^{b-1} \\
&= \frac{1}{\sqrt{2\pi}} 4^{b-1} \left(1 + O\left(\frac{1}{b}\right)\right) \left(\frac{1}{4} - \frac{y^2}{4(2b+1)}\right)^{b-1} \\
&= \frac{1}{\sqrt{2\pi}} \left(1 - \frac{y^2}{2b+1}\right)^{b-1} \left(1 + O\left(\frac{1}{b}\right)\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \left(1 + O\left(\frac{1}{b}\right)\right) \qquad -\sqrt{2b+1} < y < \sqrt{2b+1}.
\end{aligned}$$

Thus, in the limit $b \to \infty$, $f_Y(y)$ converges pointwise to the probability density function of a standard normal random variable. By Scheffé's theorem, Y converges in distribution to the standard normal distribution.