Theorem If X is a beta random variable, then $\frac{X}{1-X}$ has the inverted beta distribution.

Proof Let X be a beta random variable with parameters $\beta > 0$ and $\gamma > 0$. Then by definition, X has probability density function

$$f_X(x) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)} x^{\beta-1}(1-x)^{\gamma-1} \qquad 0 < x < 1.$$

Now consider the transformation $g(X) = \frac{X}{1-X}$, which transforms $\mathcal{X} = \{x \mid 0 < x < 1\}$ into $\mathcal{Y} = \{y \mid 0 < y < \infty\}$. The goal is to apply the transformation technique to find the probability density function of $Y = \frac{X}{1-X}$. First, $X = g^{-1}(Y) = \frac{Y}{1+Y}$. The resulting Jacobian is $\frac{dX}{dY} = \frac{1}{(1+Y)^2}$. We can then apply the transformation technique:

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\
&= f_X\left(\frac{y}{1+y}\right) \left| \frac{1}{(1+y)^2} \right| \\
&= \left[\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}\right] \left(\frac{y}{1+y}\right)^{\beta-1} \left(1 - \frac{y}{1+y}\right)^{\gamma-1} \frac{1}{(1+y)^2} \\
&= \left[\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}\right] \left(\frac{y}{1+y}\right)^{\beta-1} \left(\frac{1}{1+y}\right)^{\gamma-1} \frac{1}{(1+y)^2} \\
&= \left[\frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}\right] y^{\beta-1}(1+y)^{-\beta-\gamma} \qquad y > 0,
\end{aligned}$$

which is the probability density function of an inverted beta random variable.

APPL verification: The APPL statements

```
X := BetaRV(b, a);
g := [[x -> x / (1 - x)], [0, 1]];
Transform(X, g);
```

produce the same probability density function as found above. APPL has no built-in inverted beta distribution.