### Beta distribution

(from [http://www.math.wm.edu/~leemis/chart/UDR/UDR.html](http://www.math.wm.edu/~leemis/chart/UDR/UDR.html))

The shorthand $X \sim \text{beta}(\beta, \gamma)$ is used to indicate that the random variable $X$ has the beta distribution with parameters beta and gamma. A beta random variable $X$ with positive shape parameters $\beta$ and $\gamma$ has probability density function

$$f(x) = \frac{\Gamma(\beta + \gamma) x^{\beta - 1} (1 - x)^{\gamma - 1}}{\Gamma(\beta) \Gamma(\gamma)} \quad 0 < x < 1.$$

The beta distribution is used for modeling random variables that lie between 0 and 1 (for example, percentages or interest rates) and as a prior distribution (for example, the beta–binomial distribution). Probability density functions with various values of the parameters are illustrated below.
The cumulative distribution function on the support of $X$ is

$$F(x) = P(X \leq x) = I_x(\beta, \gamma) \quad 0 < x < 1,$$

where $I_x$ is the regularized incomplete beta function:

$$I_x(a, b) = \frac{B_x(a, b)}{B(a, b)},$$

where the beta function is

$$B(a, b) = \frac{\gamma(a) \gamma(b)}{\gamma(a + b)}$$

and the incomplete beta function is

$$B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt.$$

For integer values of $a$ and $b$, the regularized incomplete beta function can be computed via

$$I_x(a, b) = \sum_{j=a}^{a+b-1} \frac{(a + b - 1)!}{j!(a + b - 1 - j)!} x^j (1-x)^{a+b-1-j}.$$

The survivor function on the support of $X$ is

$$S(x) = P(X \geq x) = 1 - I_x(\beta, \gamma) \quad 0 < x < 1.$$

The hazard function on the support of $X$ is

$$h(x) = \frac{f(x)}{S(x)} = \frac{\Gamma(\beta + \gamma) x^{\beta-1} (1-x)^{\gamma-1}}{(1 - I_x(\beta, \gamma)) \Gamma(\beta) \Gamma(\gamma)} \quad 0 < x < 1.$$

The cumulative hazard function and the inverse distribution function are mathematically intractable. The median of $X$ is found by solving

$$I_m(\beta, \gamma) = 0.5$$

for $m$. The population mean, variance, skewness, and kurtosis of $X$ are

$$E[X] = \frac{\beta}{\beta + \gamma}$$

$$V[X] = \frac{\beta \gamma}{(\beta + \gamma)^2 (\beta + \gamma + 1)}$$

$$E \left[ \left( \frac{X - \mu}{\sigma} \right)^3 \right] = \frac{2(\gamma - \beta) \sqrt{1 + \beta + \gamma}}{\sqrt{\beta \gamma (\beta + \gamma + 2)}}$$

$$E \left[ \left( \frac{X - \mu}{\sigma} \right)^4 \right] = \frac{3(\beta^2 \gamma + \beta \gamma^2 - 2\beta \gamma + 2\beta^2 + 2\gamma^2)(1 + \beta + \gamma)}{\beta \gamma (\beta + \gamma + 2)(\beta + \gamma + 3)}$$
**APPL verification**: The APPL statements

\[
X := \text{BetaRV}(\beta, g);
\]

\[
\text{CDF}(X);
\]

\[
\text{SF}(X);
\]

\[
\text{HF}(X);
\]

\[
\text{IDF}(X);
\]

\[
\text{Mean}(X);
\]

\[
\text{Variance}(X);
\]

\[
\text{Skewness}(X);
\]

\[
\text{Kurtosis}(X);
\]

\[
\text{MGF}(X);
\]

verify the cumulative distribution function, survivor function, hazard function, population mean, variance, skewness, kurtosis, and moment generating function. Note the use of \( g \) as a parameter instead of \( \gamma \) due to APPL error.