Theorem The product of n mutually independent Bernoulli random variables is Bernoulli.

Proof Let X_1 and X_2 be independent Bernoulli random variables with parameters $0 < p_1 < 1$ and $0 < p_2 < 1$, respectively. We can write their probability mass functions as:

$$f_{X_1}(x_1) = p_1^{x_1}(1-p_1)^{1-x_1} \qquad\qquad x_1 = 0, 1$$

and

$$f_{X_2}(x_2) = p_2^{x_2}(1-p_2)^{1-x_1} \qquad\qquad x_2 = 0, 1.$$

Now consider $Y = X_1X_2$. Since both X_1 and X_2 have support $\{0, 1\}$, Y must also have this same support. So Y will be 0 when $X_1 = 0$, $X_2 = 0$, or both. So Y will be 1 when both X_1 and X_2 equal 1. This gives us the following probability mass function for Y:

$$f_Y(y) = \begin{cases} (1-p_1)p_2 + p_1(1-p_2) + (1-p_1)(1-p_2) = 1 - p_1p_2 & y = 0, \\ p_1p_2 & y = 1, \end{cases}$$

which can be rewritten as

$$f_Y(y) = p^y(1-p)^{1-y} \qquad\qquad y = 0, 1,$$

where $p = p_1p_2$. This is the probability mass function of a Bernoulli random variable with parameter p, so therefore the product of two independent Bernoulli random variables is Bernoulli.

Induction can be used with the above result to verify that the product of n mutually independent Bernoulli random variables is Bernoulli. Let $X_1, X_2, \ldots, X_n$ be n mutually independent Bernoulli random variables. Consider their product $X_1X_2\ldots X_n$. By the result above, X_1X_2 is Bernoulli. Suppose we've demonstrated that $\prod_{i=1}^{k} X_i$ is a Bernoulli random variable. Consider $\prod_{i=1}^{k+1} X_i$. Since X_{k+1} is also Bernoulli, $\prod_{i=1}^{k+1} X_i$ is Bernoulli by the result above. It follows by induction that $X_1X_2\ldots X_n$ must be Bernoulli.

APPL verification: The APPL statements

```
X1 := BernoulliRV(p1);
X2 := BernoulliRV(p2);
simplify(Product(X1, X2));
```

verify that the product of two independent Bernoulli random variables is Bernoulli.