Theorem The Benford distribution has the variate generation property, i.e., random variates can be generated in closed form via inversion.

Proof The Benford distribution has the probability mass function

$$f(x) = \log_{10}\left(1 + \frac{1}{x}\right), \qquad x = 1, 2, \ldots, 9,$$

which is equivalent to

$$f(x) = \log_{10}(x+1) - \log_{10}(x), \qquad x = 1, 2, \ldots, 9.$$

Expanding, we see that the probability mass function can be written as

$$f(x) = \begin{cases} \log_{10}(2) - 0 & x = 1 \\ \log_{10}(3) - \log_{10}(2) & x = 2 \\ \log_{10}(4) - \log_{10}(3) & x = 3 \\ \log_{10}(5) - \log_{10}(4) & x = 4 \\ \log_{10}(6) - \log_{10}(5) & x = 5 \\ \log_{10}(7) - \log_{10}(6) & x = 6 \\ \log_{10}(8) - \log_{10}(7) & x = 7 \\ \log_{10}(9) - \log_{10}(8) & x = 8 \\ \log_{10}(10) - \log_{10}(9) & x = 9. \end{cases}$$

If we generate a standard uniform variable U, and transform it by the function

$$g(x) = \lfloor 10^U \rfloor,$$

then random variates are returned according to:

$$g^{-1}(U) = \begin{cases} 1 & 0 < U < \log_{10}(2) \\ 2 & \log_{10}(2) < U < \log_{10}(3) \\ 3 & \log_{10}(3) < U < \log_{10}(4) \\ 4 & \log_{10}(4) < U < \log_{10}(5) \\ 5 & \log_{10}(5) < U < \log_{10}(6) \\ 6 & \log_{10}(6) < U < \log_{10}(7) \\ 7 & \log_{10}(7) < U < \log_{10}(8) \\ 8 & \log_{10}(8) < U < \log_{10}(9) \\ 9 & \log_{10}(9) < U < \log_{10}(10) \end{cases}$$

This yields the appropriate probabilities for $x = 1, 2, \ldots, 9$. So a variate generation algorithm is:

generate $U \sim U(0, 1)$
$X \leftarrow \lfloor 10^U \rfloor$
return(X)