Theorem The arctangent distribution has the scaling property. That is, if $X \sim \text{arctangent}(\lambda, \phi)$ then $Y = kX$ also has the arctangent distribution.

Proof Let the random variable $X$ have the arctangent($\lambda, \phi$) distribution with probability density function

$$f(x) = \frac{\lambda}{(\arctan(\lambda \phi) + \pi/2) \left(1 + \lambda^2 (x - \phi)^2\right)} \quad x \geq 0.$$ 

Let $k$ be a positive, real constant. The transformation $Y = g(X) = kX$ is a 1–1 transformation from $\mathcal{X} = \{x | x > 0\}$ to $\mathcal{Y} = \{y | y > 0\}$ with inverse $X = g^{-1}(Y) = Y/k$ and Jacobian

$$\frac{dX}{dY} = \frac{1}{k}.$$

Therefore, by the transformation technique, the probability density function of $Y$ is

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{\lambda}{(\arctan(\lambda \phi) + \pi/2) \left(1 + \lambda^2 (y/k - \phi)^2\right)} \left| \frac{1}{k} \right| = \frac{\lambda/k}{(\arctan(\lambda \phi) + \pi/2) \left(1 + (\lambda/k)^2 (y - k\phi)^2\right)} \quad x \geq 0,$$

which is the probability density function of an arctangent($\lambda/k, k\phi$) random variable.

APPL verification: The APPL statements

```appl
assume(k > 0);
X := ArcTanRV(lambda, phi);
g := [[x -> k * x], [0, infinity]];
Y := Transform(X, g);
```

yield the probability density function of an arctangent($\lambda/k, k\phi$) random variable.