Theorem The arctangent distribution has the scaling property. That is, if $X \sim \text{arctangent}(\lambda, \phi)$ then $Y = kX$ also has the arctangent distribution.

Proof Let the random variable X have the arctangent(λ, ϕ) distribution with probability density function

$$f(x) = \frac{\lambda}{\left(\arctan\left(\lambda\,\phi\right) + \pi/2\right)\left(1 + \lambda^2\left(x - \phi\right)^2\right)} \qquad x \ge 0.$$

Let k be a positive, real constant. The transformation $Y = g(X) = kX$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, x > 0\}$ to $\mathcal{Y} = \{y \,|\, y > 0\}$ with inverse $X = g^{-1}(Y) = Y/k$ and Jacobian

$$\frac{dX}{dY} = \frac{1}{k}.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| \\
&= \frac{\lambda}{\left(\arctan\left(\lambda\,\phi\right) + \pi/2\right)\left(1 + \lambda^2\left(y/k - \phi\right)^2\right)}\left|\frac{1}{k}\right| \\
&= \frac{\lambda/k}{\left(\arctan\left(\lambda\,\phi\right) + \pi/2\right)\left(1 + \left(\lambda/k\right)^2\left(y - k\phi\right)^2\right)} \qquad x \ge 0,
\end{aligned}$$

which is the probability density function of an arctangent$(\lambda/k, k\phi)$ random variable.

APPL verification: The APPL statements

```
assume(k > 0);
X := ArcTanRV(lambda, phi);
g := [[x -> k * x], [0, infinity]];
Y := Transform(X, g);
```

yield the probability density function of an arctangent$(\lambda/k, k\phi)$ random variable.