27.3 Eigenvalues and eigenvectors

As indicated at the beginning of the chapter, the scalar value λ is an eigenvalue of an $n \times n$ matrix A if there exists a nonzero vector x with n elements such that $Ax = \lambda x$. The vector x is known as the eigenvector associated with λ. The eigenvalues of A are the n solutions to the characteristic equation

$$|A - \lambda I| = 0.$$

The left-hand side of this equation is a degree n polynomial known as the characteristic polynomial. Define the matrix A as

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

The characteristic polynomial is

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5.$$

So the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0.$$

The left-hand side of this equation can be factored:

$$(\lambda - 5)(\lambda + 1) = 0.$$

Solving for λ gives the eigenvalues $\lambda = 5$ and $\lambda = -1$. Now we consider how to calculate the associated eigenvector in R. The equation $Ax = \lambda x$ for the first eigenvalue, $\lambda = 5$, is

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{aligned} x_1 + 2x_2 &= 5x_1 \\ 4x_1 + 3x_2 &= 5x_2. \end{aligned}$$

Any (x_1, x_2) pair satisfying $x_2 = 2x_1$ solves this set of equations. Hence there are an infinite number of eigenvectors associated with the eigenvalue $\lambda = 5$. The `eigen` function in R computes the eigenvalues and places them in a vector sorted in decreasing order. It also calculates the associated eigenvectors as columns of a matrix. The eigenvectors are normalized to unit length. For the matrix A, the normalized eigenvector associated with $\lambda = 5$ is $(-1/\sqrt{5}, -2/\sqrt{5})$ and the normalized eigenvector associated with $\lambda = -1$ is $(-1/\sqrt{2}, 1/\sqrt{2})$.

```
> A = matrix(c(1, 2, 4, 3), 2, 2, byrow = TRUE)
> eigen(A)              # eigenvalues and eigenvectors of A
$values
[1]  5 -1

$vectors
           [,1]       [,2]
[1,] -0.4472136 -0.7071068
[2,] -0.8944272  0.7071068
```