27.3 Eigenvalues and eigenvectors

As indicated at the beginning of the chapter, the scalar value $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if there exists a nonzero vector $x$ with $n$ elements such that $Ax = \lambda x$. The vector $x$ is known as the eigenvector associated with $\lambda$. The eigenvalues of $A$ are the $n$ solutions to the characteristic equation

$$|A - \lambda I| = 0.$$ 

The left-hand side of this equation is a degree $n$ polynomial known as the characteristic polynomial. Define the matrix $A$ as

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$ 

The characteristic polynomial is

$$|A - \lambda I| = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{vmatrix} = (1 - \lambda)(3 - \lambda) - 8 = \lambda^2 - 4\lambda - 5.$$ 

So the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0.$$ 

The left-hand side of this equation can be factored:

$$(\lambda - 5)(\lambda + 1) = 0.$$ 

Solving for $\lambda$ gives the eigenvalues $\lambda = 5$ and $\lambda = -1$. Now we consider how to calculate the associated eigenvector in R. The equation $Ax = \lambda x$ for the first eigenvalue, $\lambda = 5$, is

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$x_1 + 2x_2 = 5x_1,$$

$$4x_1 + 3x_2 = 5x_2.$$ 

Any $(x_1, x_2)$ pair satisfying $x_2 = 2x_1$ solves this set of equations. Hence there are an infinite number of eigenvectors associated with the eigenvalue $\lambda = 5$. The eigen function in R computes the eigenvalues and places them in a vector sorted in decreasing order. It also calculates the associated eigenvectors as columns of a matrix. The eigenvectors are normalized to unit length. For the matrix $A$, the normalized eigenvector associated with $\lambda = 5$ is $(-1/\sqrt{5}, -2/\sqrt{5})$ and the normalized eigenvector associated with $\lambda = -1$ is $(-1/\sqrt{2}, 1/\sqrt{2})$.

```r
> A = matrix(c(1, 2, 4, 3), 2, 2, byrow = TRUE)
> eigen(A) # eigenvalues and eigenvectors of A
$values
[1] 5 -1
$vectors
[,1] [ ,2]
[1,] -0.4472136 -0.7071068
[2,] -0.8944272 0.7071068
```