

# Survival Distributions Satisfying Benford's Law

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Hill stated that “An interesting open problem is to determine which common distributions (or mixtures thereof) satisfy Benford’s law . . .”. This article quantifies compliance with Benford’s law for several popular survival distributions. The traditional analysis of Benford’s law considers its applicability to *datasets*. This article switches the emphasis to *probability distributions* that obey Benford’s law.

KEY WORDS: Lifetimes; Random variables; Variate generation.

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## 1. INTRODUCTION

Astronomer and mathematician Simon Newcomb noticed “how much faster the first pages (of tables of logarithms) wear out than the last ones” leading to the counter-intuitive conclusion that the first significant digit in the values in a logarithm table is not uniformly distributed between 1 and 9. Using a heuristic argument, he found that ones occur most often (more than 30% of the time) and nines least often (less than 5% of the time). More specifically, if the random variable  $X$  denotes the first significant digit, then

$$\Pr(X = x) = \log_{10}(1 + 1/x),$$

for  $x = 1, 2, \dots, 9$ . He published this “logarithm law” in the *American Journal of Mathematics* in 1881.

General Electric physicist Frank Benford (1938) apparently independently arrived at the same conclusion as Newcomb concerning logarithm tables. He proceeded to “collect data from as many fields as possible” to see if natural and sociological datasets would also obey the logarithm law. He often found good agreement between the logarithm law and his 20,229 total observations, including datasets as diverse as the areas of rivers, American League baseball statistics, atomic weights of elements, death rates, and numbers appearing in *Reader's Digest*.

What has become known as “Benford’s law” has found applications in the distribution of the one-day return on

stock market indexes (Ley 1996), the distribution of the populations of 3,141 counties in the 1990 U.S. Census, and the detection of accounting fraud (Nigrini 1996).

A mathematically rigorous proof of Benford’s law has proven elusive. This is in part due to the fact that certain datasets (e.g., random numbers) do not follow Benford’s law. Recent attempts have considered the effect of scale invariance (e.g., dollars versus yen), base invariance (e.g., octal versus base ten), and mixtures (i.e., sample data drawn from several population distributions that are selected at random), as indicated in Hill (1995, 1998).

The purpose of this article is to switch the emphasis from the examination of *datasets* that obey Benford’s law to *probability distributions* that obey Benford’s law. The emphasis here is on survival distributions (i.e., random variables with positive support), although more general distributions can be examined in the same fashion.

## 2. PARAMETRIC SURVIVAL DISTRIBUTIONS

Hill (1995, pp. 361–362) stated that “An interesting open problem is to determine which common distributions (or mixtures thereof) satisfy Benford’s law . . .”. This section quantifies compliance with Benford’s law for several popular survival distributions.

As before, let  $X$  denote a random variable having Benford’s distribution, and let  $T$  denote a random lifetime with survival function  $S(t) = \Pr(T \geq t)$ . If  $Y$  is the value of the first significant digit in the lifetime  $T$ , then

$$\Pr(Y = y) = \sum_{i=-\infty}^{\infty} [S(y \cdot 10^i) - S((y+1)10^i)]$$

for  $y = 1, 2, \dots, 9$ . Thus,  $\Pr(Y = 7)$ , for example, is found by summing the appropriate probabilities on the intervals

$$\dots, (.07, .08), (.7, .8), (7, 8), (70, 80), \dots$$

More detailed examples on the derivation of the probability mass function of  $Y$  are given in Section 3.

For a particular random variable  $T$  having prescribed survivor function  $S(t)$ , it is desired to measure the goodness-of-fit between Benford’s distribution and the distribution of the first significant digit. Two such measures are the chi-square goodness-of-fit statistic

$$c = \sum_{x=1}^9 \frac{[\Pr(Y = x) - \Pr(X = x)]^2}{\Pr(X = x)},$$

and

$$m = \max_{x=1,2,\dots,9} \{|\Pr(Y = x) - \Pr(X = x)|\}.$$

These measures are calculated for several popular lifetime distributions in Table 1 as parameterized in Leemis (1995, Chapter 4).

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Table 1. Conformance to Benford's Law for Parametric Survival Distributions

Distribution	$\lambda$	$\kappa$	Class	$c$	$m$
Exponential	1		IFR/DFR	$.61 \cdot 10^{-2}$	$.29 \cdot 10^{-1}$
Exponential	5		IFR/DFR	$.54 \cdot 10^{-2}$	$.18 \cdot 10^{-1}$
Muth		.1	IFR	$.13 \cdot 10^{-1}$	$.41 \cdot 10^{-1}$
Gompertz	5	1.1	IFR	$.62 \cdot 10^{-2}$	$.20 \cdot 10^{-1}$
Weibull	1	.3	DFR	$.37 \cdot 10^{-10}$	$.16 \cdot 10^{-5}$
Weibull	1	2	IFR	.19	.11
Gamma	1	.3	DFR	$.15 \cdot 10^{-3}$	$.29 \cdot 10^{-2}$
Gamma	1	2	IFR	$.48 \cdot 10^{-1}$	$.50 \cdot 10^{-1}$
Log logistic	1	.3	DFR	$.86 \cdot 10^{-21}$	$.67 \cdot 10^{-11}$
Log logistic	1	2	UBT	$.24 \cdot 10^{-1}$	$.35 \cdot 10^{-1}$
Exponential Power Distribution	1	.3	BT	$.48 \cdot 10^{-4}$	$.17 \cdot 10^{-2}$

The following observations were made while constructing the table:

- The results for the exponential distribution for  $\lambda = 5$ , for example, are also good for  $\lambda = 5 \cdot 10^k$ , for  $k = \pm 1, \pm 2, \dots$
- For all distributions considered with a shape parameter  $\kappa$ , the goodness-of-fit measures  $c$  and  $m$  increased in  $\kappa$  for the values of  $\kappa$  considered.
- For all two-parameter distributions, the goodness-of-fit measures  $c$  and  $m$  were more sensitive to changes in the shape parameter  $\kappa$  than the scale parameter  $\lambda$ .

Notice that for the log logistic distribution with  $\lambda = 1$  and  $\kappa = .3$ , there is an astonishing 11-digit agreement with Benford's law. The fact that the probability density function of the logarithm of a log logistic random variable is symmetric might provide a clue as to why it matched Benford's law so closely.

General conditions associated with the distribution of the random variable  $T$  will now be derived in order to determine when Benford's law applies.

### 3. CONDITIONS FOR CONFORMANCE TO BENFORD'S LAW

As stated earlier, the probability density function of a Benford random variable  $X$  is

$$f_X(x) = \Pr(X = x) = \log_{10}(1 + 1/x),$$

for  $x = 1, 2, \dots, 9$ . The associated cumulative distribution function is

$$F_X(x) = \Pr(X \leq x) = \log_{10}(1 + x),$$

for  $x = 1, 2, \dots, 9$ . Inverting the cumulative distribution function, a Benford variate  $X$  can be generated by

$$X \leftarrow \lceil 10^U - 1 \rceil,$$

or

$$X \leftarrow \lfloor 10^U \rfloor,$$

where  $U \sim U(0, 1)$ .

As before, let  $T$  be the random lifetime whose first significant digit is of interest. Let the integer-valued random variable  $D$  satisfy

$$10^D \leq T < 10^{D+1}$$

(e.g.,  $T = 365 \Rightarrow D = 2$  and  $T = 1/10 \Rightarrow D = -1$ ). This definition of  $D$  allows the first significant digit  $Y$  to be written in terms of  $T$  and  $D$  as

$$Y = \lfloor T \cdot 10^{-D} \rfloor = \lfloor 10^{\log_{10} T - D} \rfloor$$

(e.g.,  $T = 365 \Rightarrow Y = \lfloor 365 \cdot 10^{-2} \rfloor = \lfloor 3.65 \rfloor = 3$ ).

Referring back to the variate generation algorithm, it is clear that if the random variable  $Z = \log_{10} T - D \sim U(0, 1)$ , which represents the result from the logarithm table, then the first significant digit  $Y$  has the Benford distribution. Using conditioning, the cumulative distribution function of  $Z$  is given by

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) \\ &= \sum_{d=-\infty}^{\infty} \Pr(10^d \leq T < 10^{d+1}) \\ &\quad \cdot \Pr(\log_{10} T - d \leq z | 10^d \leq T < 10^{d+1}), \end{aligned}$$

for  $0 < z < 1$ . Thus, conformance to Benford's law implies that the weights (the first term in the product) associated with each order of the magnitude and the distribution of  $Z = \log_{10} T - D$  (the second term in the product) are such that the infinite sum produces a linear function in  $z$ .

Why was Newcomb surprised? He expected each page of a logarithm table to be equally worn; that is, he surmised that the values that people used as arguments in logarithm tables would be uniformly distributed between 1.0 and 10.0. Although the left-hand column of a logarithm table is arranged in a linear fashion so that 1.0 to 2.0 requires  $\frac{1}{9}$  of the pages, Newcomb correctly observed that the people using the tables in 1881 did not use them in a uniform fashion (e.g., more than 30% of the table look-ups were from the first  $\frac{1}{9}$  of the pages). In summary, Newcomb expected

uniformity in the inputs to the logarithm tables, but uniformity was actually achieved in the resultant logarithms, represented by  $Z$ .

We now proceed to investigate distributions that satisfy these conditions.

*Example 1.* A distribution can be created that satisfies Benford's law exactly. Let  $W \sim U(0, 2)$ . Let  $T = 10^W$ . The probability density function of  $T$  is

$$f_T(t) = \frac{1}{2t \ln 10}$$

for  $1 < t < 100$ . The probability mass function of  $D$  is

$$\begin{aligned} f_D(0) = \Pr(D = 0) &= \Pr(1 < T < 10) \\ &= \int_1^{10} f_T(t) dt \\ &= \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} f_D(1) = \Pr(D = 1) &= \Pr(10 < T < 100) \\ &= \int_{10}^{100} f_T(t) dt = \frac{1}{2}. \end{aligned}$$

The probability mass function of the leading digit  $Y$  is

$$\begin{aligned} f_Y(y) &= \Pr(Y = y) \\ &= \Pr(y < T < y + 1) + \Pr(10y < T < 10(y + 1)) \\ &= \int_y^{y+1} f_T(t) dt + \int_{10y}^{10(y+1)} f_T(t) dt \\ &= \int_y^{y+1} \frac{1}{2t \ln 10} dt + \int_{10y}^{10(y+1)} \frac{1}{2t \ln 10} dt \\ &= \log_{10} \left( \frac{y + 1}{y} \right) \end{aligned}$$

for  $y = 1, 2, \dots, 9$ . This probability mass function matches Benford's distribution exactly.

Alternatively, one can proceed by determining the distribution of  $Z = \log_{10} T - D$ , where  $W = \log_{10} T$ .

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) \\ &= \sum_{d=-\infty}^{\infty} \Pr(10^d \leq T < 10^{d+1}) \\ &\quad \cdot \Pr(\log_{10} T - d \leq z | 10^d \leq T < 10^{d+1}) \\ &= \sum_{d=0}^1 \Pr(d \leq W < d + 1) \\ &\quad \cdot \Pr(W - d \leq z | d \leq W < d + 1) \\ &= \Pr(0 \leq W < 1) \cdot \Pr(W \leq z | 0 \leq W < 1) \\ &\quad + \Pr(1 \leq W < 2) \cdot \Pr(W - 1 \leq z | 1 \leq W < 2) \\ &= \frac{1}{2}z + \frac{1}{2}z \\ &= z \end{aligned}$$

for  $0 < z < 1$ . Since this is the cumulative distribution function for a  $U(0, 1)$  random variable, Benford's law is satisfied exactly.

The previous example can be generalized as follows. Let  $W \sim U(a, b)$ , where  $a$  and  $b$  are real numbers satisfying  $a < b$ . If the interval  $(10^a, 10^b)$  covers an integer number of orders of magnitude, then the first significant digit of the random variable  $T = 10^W$  satisfies Benford's law exactly. Equivalently, if  $b - a$  is a positive integer, then the first significant digit of  $T = 10^W$  satisfies Benford's law. Examples include  $a = -2, b = 1$  and  $a = \log_{10}(3/2), b = \log_{10}(150)$ .

There is no need for the support of the distribution of  $T = 10^W$  to span several orders of magnitude as is the case for many of the datasets that conform to Benford's law. Example 1 shows that a single order of magnitude (e.g.,  $a = 5, b = 6$ ) is sufficient.

Example 2 considers a nonuniform distribution for  $W$ .

*Example 2.* Let  $W \sim \text{Triangular}(0, 1, 2)$ . The probability density function for  $W$  is

$$f_W(w) = \begin{cases} w & 0 < w < 1 \\ 2 - w & 1 \leq w < 2. \end{cases}$$

As before, let  $T = 10^W$  and  $Z = W - D$ . The cumulative distribution function of  $Z$  is

$$\begin{aligned} F_Z(z) &= \sum_{d=0}^1 \Pr(d \leq W < d + 1) \\ &\quad \cdot \Pr(W - d \leq z | d \leq W < d + 1) \\ &= \frac{1}{2}z^2 + \frac{1}{2}(2z - z^2) \\ &= z \end{aligned}$$

for  $0 < z < 1$ . Thus, the first significant digit of  $T$  satisfies Benford's law exactly.

This example can also be generalized. Let  $W \sim \text{Triangular}(a, b, c)$ , where  $a, b$ , and  $c$  are real numbers satisfying  $a < b < c$ . The first significant digit of the random variable  $T = 10^W$  satisfies Benford's law exactly if  $a, b$ , and  $c$  are integers.

The symmetric, integer-parameter triangular distribution's conformance to Benford's law may provide some insight into the log logistic's stellar performance in Table 1. If the probability density function of  $W$  is symmetric about an integer and the variance of  $W$  is large, then it is often the case that the probability density function of  $W$  is approximately linear between consecutive integers. The symmetric portions of the probability density function of  $W$  will nearly cancel one another when computing the distribution of  $Z$ . A normal random variable  $W$  with integer mean  $\mu$  and large standard deviation  $\sigma$ , for example, corresponds to a lognormal  $T = 10^W$  whose first significant digit closely approximates Benford's law.

Example 3 considers a nonsymmetric distribution for  $W$ .

*Example 3.* Let  $W$  have probability density function

$$f_W(w) = \begin{cases} 1 - w^2 & -1 < w < 0 \\ (w - 1)^2 & 0 \leq w < 1. \end{cases}$$

As before, let  $T = 10^W$  and  $Z = W - D$ . The cumulative distribution function of  $Z$  is

$$\begin{aligned} F_Z(z) &= \sum_{d=-1}^0 \Pr(d \leq W < d+1) \\ &\quad \cdot \Pr(W - d \leq z | d \leq W < d+1) \\ &= \frac{2}{3} \left[ -\frac{z^3}{2} + \frac{3z^2}{2} \right] + \frac{1}{3} [1 + (z-1)^3] \\ &= z \end{aligned}$$

for  $0 < z < 1$ . Thus, the first significant digit of  $T$  satisfies Benford's law exactly.

This example can be generalized for  $W$  with probability density function

$$f_W(w) = \begin{cases} 1 - w^n & -1 < w < 0 \\ (w - 1)^n & 0 \leq w < 1, \end{cases}$$

where  $n$  is a positive, even integer.

We wanted to experiment with several other probability distributions in order to evaluate conformance to Benford's law. In order to automate this process, we wrote an APPL (Glen, Leemis, and Evans in press) procedure `Benford`, whose argument is the distribution of  $W$  and whose returned value is the distribution of  $Z$ . The algorithm is shown in Figure 1.

The code required to return the distribution of  $Z$  for the triangular distribution in Example 2, for instance, is

```
W := TriangularRV(0, 1, 2);
Z := Benford(W);
```

After experimenting with `Benford` on other distributions, we have come to the following conclusions:

1. Distributions of  $W$  with a single mode that occurs at either extreme of their support will never satisfy Benford's law (e.g.,  $W \sim$  exponential).

2. Using a geometric argument, certain limiting distributions of  $W$  (e.g.,  $W \sim N(\mu, \sigma^2)$ , where  $\mu$  is an integer and  $\sigma \rightarrow \infty$ ) will satisfy Benford's law.

3. Other distributions (e.g., Weibull) may come very close to satisfying Benford's law for various parameter values. Our experimentation revealed that compliance with Benford's law depends on parameter values within one particular parametric family. Thus, using Benford's law to detect accounting fraud, for example, is dubious due to an unacceptably high rate of false positives.

4. For a random variable  $T$  that can assume negative values, all of the results given here apply since the first digit of  $|T|$  equals the first digit of  $T$ .

5. If  $W$  is a distribution such that the first significant digit of  $10^W$  satisfies Benford's law, then the first significant digit of  $b^W$  satisfies Benford's law for base  $b = 2, 3, \dots$

6. The distribution associated with the more general form of Benford's law

$$\Pr(\text{mantissa} < t) = \log_{10} t \quad 1 \leq t < 10,$$

where the mantissa of a real number is the number obtained from shifting the decimal point to the place immediately following the first significant (nonzero) digit, is *sum-invariant* (Allaart 1997). A short proof of a generalization of Allaart's result appears below.

*Result.* Using our earlier notation, let  $W \sim U(0, 1)$  and  $T = 10^W$ . Then the random variable  $T$  with cdf,  $F_T(t) = \log_{10} t$  for  $1 \leq t < 10$ , is sum-invariant; that is, if the interval  $[1, 10)$  is equally partitioned by  $h > 0$ , then the expected sum of variates in any given partitioned interval is the same.

*Proof.* Let  $k$  be any natural number and set  $h = \frac{9}{k}$ . Without loss of generality, fix  $k$ . Let  $A_j = [1 + (j-1) \cdot h, 1 + j \cdot h) \subset \mathcal{R}$  for  $j = 1, 2, \dots, k$ . The probability that  $T$  is in the interval  $A_j$  and the conditional expected value of  $T$  on the interval  $A_j$  for any  $j = 1, 2, \dots, k$  are, respectively,

$$\begin{aligned} &\Pr(1 + (j-1) \cdot h \leq T < 1 + j \cdot h) \\ &= \int_{1+(j-1) \cdot h}^{1+j \cdot h} \frac{1}{x \ln(10)} dx \\ &= \frac{\ln(1 + j \cdot h) - \ln(1 + (j-1) \cdot h)}{\ln(10)}, \end{aligned}$$

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<pre> Ω ← Support(W) Lo ← ⌊Ω⌋ Hi ← ⌈Ω⌉ - 1 Weight ← Array[1..Hi - Lo + 1] TransfW ← Array[1..Hi - Lo + 1]  For d ← Lo to Hi   Weight[d] ← F<sub>W</sub>(d + 1) - F<sub>W</sub>(d)   TruncW[d] ← Truncate(W, d, d + 1)   TransfW[d] ← Transform(TruncW, w - d) Z ← Mixture(Weight, TransfW) </pre>	<p>[The set <math>\Omega</math> is the support of the random variable <math>W</math>]</p> <p>[Lower loop limit]</p> <p>[Upper loop limit]</p> <p>[Let Weight hold the mixture probabilities]</p> <p>[Let TransfW hold the transformed segments of <math>W</math>]</p> <p>[Calculate weights for the mixture]</p> <p>[Truncate <math>W</math> between <math>d</math> and <math>d + 1</math>]</p> <p>[Horizontally shift <math>W</math> by <math>d</math> units]</p> <p>[Compute the distribution of the mixture]</p>
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Figure 1. Algorithm for procedure `Benford`.

and

$$\begin{aligned} E(T|1 + (j-1) \cdot h \leq T < 1 + j \cdot h) \\ &= \int_{1+(j-1) \cdot h}^{1+j \cdot h} x \cdot \frac{1}{x(\ln(1 + j \cdot h) - \ln(1 + (j-1) \cdot h))} dx \\ &= \frac{h}{\ln(1 + j \cdot h) - \ln(1 + (j-1) \cdot h)}. \end{aligned}$$

Thus, the expected sum of  $n$  variates in the interval  $A_j$  for any  $j$  is

$$n \cdot E(T|1 + (j-1) \cdot h \leq T < 1 + j \cdot h) \cdot \Pr(1 + (j-1) \cdot h \leq T < 1 + j \cdot h) = \frac{n \cdot h}{\ln(10)},$$

Since this expected sum depends on  $k$  and is independent of  $j$ , the distribution of  $T$  is sum-invariant.

7. Any mixture of distributions that individually obey Benford's law will obey Benford's law. The case of two random variables satisfying Benford's law is proven below.

*Result.* Let  $T_1$  and  $T_2$  be nonnegative random variables whose first significant digits satisfy Benford's law. Let the random variable  $T$  have probability density function

$$f_T(t) = pf_{T_1}(t) + (1-p)f_{T_2}(t) \quad t > 0,$$

for  $0 < p < 1$ . Then  $T$  also satisfies Benford's law.

*Proof.* Let  $Z_1 = \log_{10} T_1 - D$ ,  $Z_2 = \log_{10} T_2 - D$ , and  $Z = \log_{10} T - D$ . Since  $T_1$  and  $T_2$  satisfy Benford's law, then  $F_{Z_1}(z) = z$  and  $F_{Z_2}(z) = z$ . In order to prove that  $T$  also satisfies Benford's law, we need to show that  $F_Z(z) = z$ . By conditioning on  $z$ , we have

$$\begin{aligned} F_Z(z) &= pF_{Z_1}(z) + (1-p)F_{Z_2}(z) \\ &= pz + (1-p)z \\ &= z \end{aligned}$$

for  $0 < z < 1$ .

#### 4. VARIATE GENERATION

As stated earlier, variates from the Benford distribution can be generated via

$$X \leftarrow \lfloor 10^U \rfloor,$$

where  $U \sim U(0, 1)$ . Two variations of this algorithm can be developed by allowing different bases and multiple significant digits as described in the next two paragraphs.

Benford's law for the first significant digit in base  $b$  is associated with the probability density function

$$f_X(x) = \Pr(X = x) = \log_b(1 + 1/x),$$

for  $x = 1, 2, \dots, b-1$  and  $b = 2, 3, \dots$ . Since the cumulative distribution function is

$$F_X(x) = \Pr(X \leq x) = \log_b(1 + x),$$

for  $x = 1, 2, \dots, b-1$ , variates can be generated via

$$X \leftarrow \lfloor b^U \rfloor,$$

where  $U \sim U(0, 1)$ . Note that when  $b$  is 2 (the binary case), the  $X$  value generated is always 1, as expected.

When the first  $r$  digits are considered, Benford's law generalizes to

$$f_X(x) = \Pr(X = x) = \log_{10}(1 + 1/x),$$

for  $x = 10^{r-1}, 10^{r-1} + 1, \dots, 10^r - 1$ , for  $r = 1, 2, \dots$ . [Note that this rather relaxed notation implies that  $x = 365$  when  $r = 3$  corresponds to a first digit  $R_1 = 3$ , second digit  $R_2 = 6$ , and third digit  $R_3 = 5$ , which occurs with probability  $\Pr(X = 365) = \Pr(R_1 = 3, R_2 = 6, R_3 = 5) = \log_{10}(1 + 1/365)$ .] The cumulative distribution function is

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) \\ &= \sum_{i=10^{r-1}}^x \log_{10}(1 + 1/i) \\ &= \sum_{i=10^{r-1}}^x \log_{10}(i + 1) - \log_{10} i \\ &= \log_{10}(x + 1) - \log_{10}(10^{r-1}) \\ &= \log_{10}\left(\frac{x + 1}{10^{r-1}}\right) \end{aligned}$$

for  $x = 10^{r-1}, 10^{r-1} + 1, \dots, 10^r - 1$ . Variates can be generated by inversion via

$$X \leftarrow \lfloor 10^{U-r+1} \rfloor,$$

where  $U \sim U(0, 1)$ .

Combining the previous two cases, a discrete Benford variate  $X$  associated with the first  $r$  significant digits in base  $b$  is generated by inversion via

$$X \leftarrow \lfloor b^{U-r+1} \rfloor,$$

where  $U \sim U(0, 1)$ .

#### 5. CONCLUSIONS

Benford's law holds exactly for certain parametric survival distributions introduced in Section 3, holds to varying degrees for many other parametric distributions as shown in Section 2, and holds very poorly [e.g.,  $T \sim U(3, 7)$  since the digits 1, 2, 7, 8, 9 never occur or the number of children in a family in the U.S.] for other distributions. The reason that Benford's law applies to so many datasets may simply be due to the fact that many popular parametric lifetime models also closely follow his law for particular values of their parameters.

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