Variate Generation for the Accelerated Life and Proportional Hazards Models

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ABSTRACT

The accelerated life and proportional hazards lifetime models are used to account for the effects of covariates on a random lifetime. Variate generation algorithms for Monte Carlo simulation in both the renewal and nonhomogeneous Poisson process cases are a simple extension of the inverse-cdf technique.

Introduction

The effect of covariates on survival often complicates the analysis of a set of lifetime data. In a medical setting, these covariates are usually patient characteristics such as age, gender or blood pressure. In reliability, covariates such as the turning speed of a machine tool or the stress applied to a component affect the lifetime of an item. Two models that are often used to incorporate the effect of these covariates on lifetimes are the *accelerated life* and *proportional hazards* models. The purpose of this technical note is to give algorithms for the generation of lifetimes for Monte Carlo simulation which are described by one of these models.

Devroye (1986a) has written the first textbook completely devoted to variate generation. Although this paper considers only the inverse-cdf technique, more complicated techniques such as thinning, developed by Lewis and Shedler (1979) and illustrated in Ogata (1981), may be adapted to variate generation for models incorporating covariates. Schmeiser (1980) reviews variate generation techniques and Devroye (1986b) compares the speed of several random variate generation algorithms.

1. Notation

Let $h(t)$ and $H(t) = \int_0^t h(\tau)d\tau$ be the hazard and cumulative hazard functions, respectively, for a continuous nonnegative random variable T, the lifetime of the item under study. The $q \times 1$ vector $\mathbf{z}$ contains covariates associated with a particular item or individual. The covariates are linked to the lifetime by the function $\psi(\mathbf{z})$, which satisfies

$\psi(\mathbf{O}) = 1$ and $\psi(\mathbf{z}) \geq 0$ for all $\mathbf{z}$. A popular choice is $\psi(\mathbf{z}) = e^{\beta' \mathbf{z}}$, where β is a $q \times 1$ vector of regression coefficients.

The cumulative hazard function for T in the *accelerated life* model is (Cox and Oakes, 1984)

$$H(t) = H_0(t\,\psi(\mathbf{z}))$$

where H_0 is a baseline cumulative hazard function. Note that when $\mathbf{z} = \mathbf{O}$, $H_0 \equiv H$. In this model, the covariates accelerate ($\psi(\mathbf{z}) > 1$) or decelerate ($\psi(\mathbf{z}) < 1$) the rate at which the item moves through time. The *proportional hazards* model

$$H(t) = \psi(\mathbf{z})\, H_0(t)$$

increases ($\psi(\mathbf{z}) > 1$) or decreases ($\psi(\mathbf{z}) < 1$) the failure rate of the item by the factor $\psi(\mathbf{z})$ for all values of t.

2. Variate Generation Algorithms

Griffith (1982) and others have observed that the cumulative hazard function, $H(T)$, has a unit exponential distribution. Therefore, a random variate t corresponding to a cumulative hazard function $H(t)$ can be generated by

$$t = H^{-1}(-\log(u))$$

where u is uniformly distributed between 0 and 1. In the accelerated life model, since time is being expanded or contracted by a factor $\psi(\mathbf{z})$, variates are generated by

$$t = \frac{H_0^{-1}(-\log(u))}{\psi(\mathbf{z})}\ .$$

In the proportional hazards model, equating $-\mathit{log}(u)$ to $H(t)$ yields the variate generation formula

$$t = H_0^{-1}\left(\frac{-\log(u)}{\psi(\mathbf{z})}\right).$$

In addition to generating individual lifetimes, these variate generation techniques may also be applied to point processes. A renewal process, for example, with time between events having a cumulative hazard function $H(t)$, can be simulated by using the appropriate generation formula for the two cases shown above. These variate generation formulas must be modified, however, to generate variates from a nonhomogeneous Poisson process (NHPP).

In a NHPP, the hazard function, $h(t)$, is equivalent to the intensity function, which governs the rate at which events occur. To determine the appropriate method for generating variates from a NHPP, assume that the last event in a point process has occurred at time a. The cumulative hazard function for the time of the next event conditioned on survival to time a is

$$H_{T \mid T > a}(t) = H(t) - H(a) \qquad\qquad t > a$$

In the accelerated life model, where $H(t) = H_0(t\,\psi(\mathbf{z}))$, the time of the next event is generated by Equating the conditional cumulative hazard function to $-log(u)$, the time of the next event in the proportional hazards case is generated by

$$t = H_0^{-1}\left(H_0(a) - \frac{log(u)}{\psi(\mathbf{z})}\right).$$

3. Example

The exponential power distribution (Smith and Bain, 1975) is a flexible two parameter distribution with cumulative hazard function

$$H(t) = e^{(t/\alpha)^{\gamma}} - 1 \qquad\qquad \alpha > 0,\ \gamma > 0,\ t > 0$$

and inverse cumulative hazard function

$$H^{-1}(y) = \alpha[log(y+1)]^{1/\gamma}.$$

Assume that the covariates are linked to survival by the function $\psi(\mathbf{z}) = e^{\beta'\mathbf{z}}$ in the accelerated life model. If a NHPP is to be simulated, the baseline hazard function has the exponential power distribution with parameters α and γ, and the previous event has occurred at time a, then the next event is generated at time

$$t = \alpha e^{-\beta'\mathbf{z}}[log(e^{(ae^{\beta'\mathbf{z}}/\alpha)^{\gamma}} - log(u))]^{1/\gamma}$$

where u is uniformly distributed between 0 and 1.

4. Conclusions

As survival models in reliability and biostatistics become more complex, Monte Carlo simulation will be relied on more often to estimate measures of performance due to tractability problems. This technical note has shown that in addition to being tractable models for incorporating covariates into a lifetime model, the accelerated life and proportional hazards models do not substantially complicate variate generation. The table below gives formulas for generating event times from a renewal or nonhomogeneous Poisson process, where the last event is assumed to have occurred at time a:

	Renewal	NHPP
Accelerated life	$t = a + \frac{H_0^{-1}(-\,log(u))}{\psi(\mathbf{z})}$	$t = \frac{H_0^{-1}(H_0(a\psi(\mathbf{z})) - \,log(u))}{\psi(\mathbf{z})}$
Proportional hazards	$t = a + H_0^{-1}(\frac{-\,log(u)}{\psi(\mathbf{z})})$	$t = H_0^{-1}(H_0(a) - \frac{log(u)}{\psi(\mathbf{z})})$

The renewal and NHPP algorithms are equivalent when $a = 0$ (since a renewal process is equivalent to a NHPP restarted at zero after each event), the accelerated life and proportional hazards models are equivalent when $\psi(\mathbf{z}) = 1$, and all four cases shown in the above table are equivalent when $H_0(t) = \lambda t$ (the exponential case) because of the memoryless property.

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