Lifetime Distribution Identities

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Key Words—**Mean residual life, identities for lifetime distributions**

Reader Aids—
Purpose: Tutorial
Special math needed for explanations: Elementary probability theory
Special math needed to use results: Same
Results useful to: Reliability beginners

Abstract—**Five ways of representing the distribution of a continuous nonnegative random variable T are used extensively in the reliability literature: the probability density function, the reliability (survivor function), the hazard rate, the cumulative hazard function, and the mean residual life function. Properties, identities, and intuitive interpretations of the five representations are discussed. Several examples are given.**

Although there are other functions, such as normalized mean residual life for studying replacement policies, these five distribution representations have surfaced as vehicles for representing a lifetime distribution. The choice of which distribution representation to use depends on whether —

1. **The representation has a tractable form**
2. **Intuition is gained concerning the distribution by seeing a plot of the representation.**

1. INTRODUCTION

In reliability, a continuous nonnegative random variable typically represents the lifetime of an element. There are several functions which completely specify the distribution of the random variable. Examples of these functions include the probability density function, characteristic function, Mellin transform, and cumulative distribution function. Five mathematically equivalent, popular representations have evolved: the probability density function, the reliability, the hazard rate, the cumulative hazard function, and the mean residual life function. Each of these functions completely describes the distribution of a lifetime, and any one of the functions determines the other four. This paper examines the reasons why these five have emerged as popular ways to describe a lifetime distribution in the reliability literature, and summarizes their useful properties and identities.

2. ASSUMPTIONS, NOMENCLATURE AND NOTATION

Discussion is limited to describing properties of *populations* of items or components, as opposed to properties of samples. Thus, the distribution representations refer to mathematical properties of random variables, not statistical inference on samples. Also, discussion is limited to *nonrepairable* items such as fuses or lightbulbs, as opposed to items which can be renewed via preventive maintenance or repair. The origins of the use of these distribution representations and their applications date back to the 1950's in papers such as Epstein & Sobel [4].

These five representations are not the only possible ways to represent a nonnegative random variable T. Other representations include the moment generating function $\mathrm{E}\{e^{sT}\}$, the characteristic function $\mathrm{E}\{e^{isT}\}$, the Mellin transform $\mathrm{E}\{T^s\}$, the density quantile function $f(F^{-1}(u))$, and the total time on test transform $\int_0^{F^{-1}(t)} R(x)dx$, for $0 \le t \le 1$, where F^{-1} is the inverse-Cdf. The density quantile function is discussed in Parzen [10], and the total time on test transform is discussed in Barlow [1] and Gupta & Michalek [6].

Each of the distribution representations described in sections 3-7 follow similar presentations. First, the name of the representation is given, along with any other names the representation is known by (eg, force of mortality for the hazard rate). Second, conditions for existence are given. Finally, a list of properties, intuitive interpretations, and definitions associated with the representations are listed.

Notation

T	a continuous nonnegative random variable
μ	$\mathrm{E}\{T\}$
$\sim$	implies "is distributed as"
$f(t)$	pdf for T
$R(t)$	reliability (survivor function) for T
$h(t)$	hazard rate for T
$H(t)$	cumulative hazard function for T
$L(t)$	mean residual life for T
$U(0, 1)$	uniform distribution on [0, 1]

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

Acronyms

IFR	Increasing Failure Rate
DFR	Decreasing Failure Rate
NBU	New Better than Used
NWU	New Worse than Used
NBUE	New Better than Used in Expectation
NWUE	New Worse than Used in Expectation
IMRL	Increasing Mean Residual Life
DMRL	Decreasing Mean Residual Life
i.i.d.	statistically independent and identically distributed

3. PROBABILITY DENSITY FUNCTION

The pdf satisfies —

$f(t) \geq 0$, for all t

$\int_0^\infty f(t)dt = 1.$

The pdf is useful because the probability of failure between times a and b is the integral of $f(t)$ between a and b. Properties of the pdf include:

1. $f(t)\Delta t = \Pr\{t < T < t + \Delta t\}$ for small Δt.
2. If $f(t)$ has only one mode which occurs at the origin, the distribution is DFR (Watson & Wells [13]).
3. $f(t_i)$ is the appropriate factor in the likelihood function for an uncensored data value t_i, eg, a failure time.
4. Finite mixture models for m populations of components may be modeled using the pdf:

$f(t) = \sum_{i=1}^{m} p_i f_i(t)$, where $f_i(t)$ is the pdf for population i,

and p_i is the probability of selecting a component from population i, $i = 1, \ldots, m$.

4. RELIABILITY (SURVIVOR FUNCTION)

The reliability (also known as the survivor function and complementary Cdf) is defined by —

$R(t) = \Pr\{T > t\} = \int_t^\infty f(\tau)d\tau$

which is a nonincreasing function of t satisfying $R(0) = 1$ and $\lim_{t \to \infty} R(t) = 0$. The reliability is important in the study of *systems* of components since it is the appropriate argument in the structure function to determine system reliability. Properties of $R(t)$ include:

1. $R(t)$ is the fraction of the population which will survive to time t. It is also the probability that one single item will survive to time t.
2. $R(t)$ is uniformly distributed between zero and one by the probability integral transformation. Thus, $R^{-1}(U)$ generates a lifetime variate for Monte Carlo simulation where U is distributed as $U(0, 1)$.
3. $R(t_i)$ is the appropriate factor in the likelihood function for a right censored (ie, only a lower bound on the failure time is known) data value t_i.
4. The NBU and NWU distribution classes are easily defined in terms of $R(t)$. Barlow & Proschan [2] define these distribution classes, as well as other distribution classes cited in this paper. A distribution is NBU if and only if —

$R(x + y) \leq R(x)R(y)$ for $x \geq 0$ and $y \geq 0$;

and NWU if and only if —

$R(x + y) \leq R(x)R(y)$ for $x \geq 0$ and $y \geq 0$.

5. HAZARD RATE

The hazard rate (also known as the rate function, intensity function, force of mortality) can be defined by —

$h(t) = f(t)/R(t)$

and satisfies —

$h(t) \geq 0$ for all t and $\int_0^\infty h(\tau)d\tau = \infty$.

The reciprocal of the hazard function is also known as Mill's ratio. The hazard rate is popular in reliability work because it has the intuitive interpretation as the amount of *risk* associated with a component which has survived to time t. Properties of $h(t)$ include:

1. $h(t)\Delta t = \Pr\{t < T < t + \Delta t \mid T > t\}$ for small Δt.
2. $h(t)$ is a special form of the complete intensity function at time t for a point process (Cox & Oakes [3]). In other words, the hazard function is mathematically equivalent to the intensity function for a nonhomogeneous Poisson process, and the failure time corresponds to the first event in the process.
3. The IFR and DFR distribution classes are easily defined in terms of $h(t)$. A distribution is IFR if and only if $h(t)$ is nondecreasing in t. A distribution is DFR if and only if $h(t)$ is nonincreasing in t.
4. Competing risks models are easily formulated in terms of $h(t)$. If $h_1(t), h_2(t), \ldots, h_k(t)$ are the k causes of failure acting in a population, and $T = \min\{T_1, T_2, \ldots, T_k\}$, then the hazard function for the time to failure is $h(t) = \sum_{j=1}^{k} h_j(t)$.
5. When the event of concern is failure, $h(t) = \lambda(t)$ is the failure rate.

6. CUMULATIVE HAZARD FUNCTION

The cumulative hazard function can be defined by —

$H(t) = \int_0^t h(\tau)d\tau$

and is a nondecreasing function of time which satisfies: $H(0) = 0$ and $\lim_{t \to \infty} H(t) = \infty$. The cumulative hazard function has utility in reliability work because of the following properties:

1. Since $H(t) = -\log R(t)$, $H(t)$ has an exponential distribution with a mean of one. Thus, $H^{-1}(-\log(1 - U))$ generates a variate for Monte Carlo simulation when U is distributed as $U(0, 1)$, as shown in Griffith [5].
2. The cumulative hazard function parallels the renewal function from renewal theory. Renewal theory, as described in Karlin & Taylor [7], examines a process with

i.i.d. times between events. The renewal function is the mean number of events by time t.

3. The IFRA and DFRA distribution classes are easily defined in terms of $H(t)$. A distribution is IFRA if and only if $H(t)/t$ is nondecreasing in t. A distribution is DFRA if and only if $H(t)/t$ is nonincreasing in t.

7. MEAN RESIDUAL LIFE

The mean residual life can be defined by —

$$L(t) = \int_t^\infty \frac{R(\tau)d\tau}{R(t)}$$

and satisfies the three conditions given by Swartz [12]:

$$L(t) \geq 0, \frac{dL(t)}{dt} \geq -1, \int_0^\infty \frac{1}{L(t)} dt = \infty.$$

Properties of the mean residual life include:

1. $L(t)$ has the intuitive interpretation as the mean remaining lifetime of an item which has survived to time t, ie, $L(t) = \mathrm{E}\{T - t | T > t\}$.
2. $\mu = \mathrm{E}\{T\} = L(0)$; μ is the mean lifetime of the unit.
3. Burn-in and replacement models often make decisions based on the ratio of $L(t)$ to $L(0)$, as indicated in Watson & Wells [13] and Muth [9].
4. The IMRL, DMRL, NBUE and NWUE classes are easily defined in terms of $L(t)$. A distribution is IMRL if $L(t)$ is nondecreasing in t. A distribution is NBUE if

$L(t) \geq \mu$ for all t.

A distribution is DMRL if $L(t)$ is nonincreasing in t. A distribution is NWUE if

$L(t) \leq \mu$ for all t.

Mantel [8] gives an ordering of distributions based on their mean residual life in the right hand tail. The mean residual life has special meaning for *heavy tailed* distributions with low failure rates. If a distribution has an exponential tail, for example, with a failure rate of 10^{-6} per hour, the mean residual life value in the tail is 100 years! Thus, the chances are rather slim of observing a failure once the component survives to the tail of the lifetime distribution.

8. IDENTITIES

The five distribution representations defined in sections 3-7 are related to one another by (where a prime implies the first derivative with respect to t)

1. $f(t) = -R'(t)$
2. $H(t) = -\log R(t)$
3. $h(t) = H'(t)$
4. $L(t) = \int_t^\infty \frac{R(\tau)}{R(t)} d\tau$
5. $h(t)\, L(t) = 1 + L'(t)$

Any one of the distribution representations implies the other four, parallel to trigonometric identities. For example, if the second identity is differentiated,

$$h(t) = -\frac{d}{dt} \log R(t) = \frac{f(t)}{R(t)}.$$

If both sides of the second identity are exponentiated,

$$R(t) = e^{-\int_0^t h(\tau)d\tau}.$$

Thus, from these five identities, and the definitions of each representation, many other identities can be calculated.

9. EXAMPLES

Four examples of the distribution representations are illustrated in this section: the exponential, Weibull, gamma, and the exponential power distributions. The exponential, Weibull, and gamma are surveyed in Barlow & Proschan [2], and the exponential power distribution is given in Smith & Bain [11]. Figure 1 shows $f(t)$, $R(t)$, $h(t)$, $H(t)$, $L(t)$ for various parameter values for all four distributions with a scale parameter of unity. In all of the distributions, α is a positive scale parameter and β is a positive shape parameter.

9.1 The exponential distribution has a single scale parameter α, and distribution representations are all closed form:

$$f(t) = \alpha e^{-\alpha t}, R(t) = e^{-\alpha t}, h(t) = \alpha, H(t) = \alpha t, L(t) = \frac{1}{\alpha}$$

The exponential distribution is the only distribution which belongs to all of the distribution classes (eg, IFR, DFR, IFRA) mentioned in this paper.

9.2 The Weibull and gamma distributions are both generalizations of the exponential distribution, with pdfs:

$$f(t) = \alpha\beta t^{\beta-1} e^{-\alpha t^\beta}$$

and

$$f(t) = \frac{\alpha^\beta}{\Gamma(\beta)} t^{\beta-1} e^{-\alpha t}$$

respectively. Both are IFR when $\beta > 1$ and DFR when $\beta < 1$.

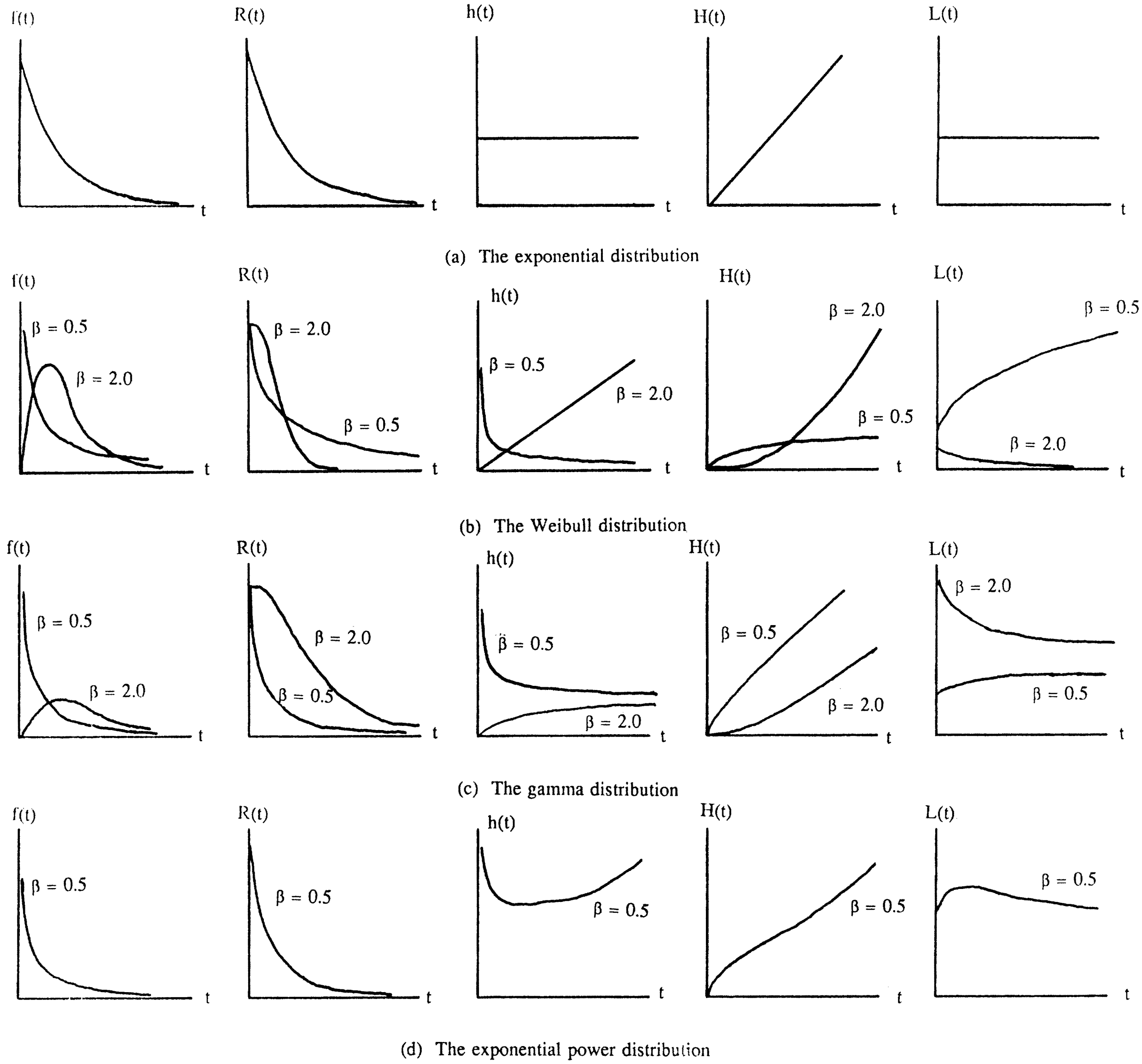

Fig. 1. Distribution representations

The graph of $h(t)$ shows that all gamma distributions have exponential right hand tails, since $\lim_{t \to \infty} h(t) = \alpha$ for all β values.

9.3 The exponential power distribution is one of the few tractable two parameter distributions which can achieve a bathtub shaped hazard rate:

$$h(t) = \alpha\beta t^{\beta-1} e^{\alpha t^{\beta}}.$$

When $\beta < 1$, a hazard function achieves a minimum at $\left(\frac{1-\beta}{\alpha\beta}\right)^{1/\beta}$. Correspondingly, the mean residual life function increases initially (ie, after surviving infant mortality), then decreases.

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REFERENCES

[1] R. E. Barlow, "Geometry of the total time on test transform", *Naval Research Logistics Quarterly,* vol 26, 1979, pp 393-402.

[2] R. E. Barlow, F. Proschan, *Statistical Theory of Reliability and Life Testing,* 1981. Available from: TO BEGIN WITH, c/o Gordon Pledger; 1142 Hornell Drive; Silver Spring, MD 20904, USA.

[3] D. R. Cox, D. Oakes, *Analysis of Survival Data,* Chapman & Hall, 1984.

[4] B. Epstein, M. Sobel, "Life testing", *J. Amer. Statistical Assoc.,* vol 48, no. 263, 1953, pp 486-502.

[5] W. S. Griffith, "Representations of distributions having monotone or bathtub-shaped failure rate", *IEEE Trans. Reliability,* vol R-31, 1982 Apr, pp 95-96.

[6] R. C. Gupta, J. E. Michalek, "Determination of reliability functions by the TTT transform", Technical Report, University of Maine, Orono.

[7] S. Karlin, H. M. Taylor, *A First Course in Stochastic Processes*, Second Edition, Academic Press, 1975.
[8] N. Mantel, "Tails of distributions", *The Amer. Statistician*, vol 30, 1976 Feb, pp 14-17.
[9] E. J. Muth, "Memory as a property of probability distributions", *IEEE Trans. Reliability*, vol R-29, 1980 Jun, pp 160-164.
[10] E. Parzen, "Nonparametric statistical data modeling", *J. Amer. Statistical Assoc.*, vol 74, no. 365, 1979 Mar, pp 105-131.
[11] R. M. Smith, L. J. Bain, "An exponential power life-test distribution", *Communications in Statistics*, vol 4, no. 5, 1975, pp 469-481.
[12] G. B. Swartz, "The mean residual life function", *IEEE Trans. Reliability*, vol R-22, 1973 Jun, pp 108-109.
[13] G. S. Watson, W. T. Wells, "On the possibility of improving the mean useful life of items by eliminating those with short lives", *Technometrics*, vol 3, 1961 May, pp 281-298.

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CORRECTIONS *1980 APRIL Issue* *CORRECTIONS* *1980 APRIL Issue* *CORRECTIONS*

Correction to: Age Replacement Policies for Weibull Failure Times

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Key Words—**Age replacement, Optimal policy, Weibull distribution.**

Abstract—**Table 1 in the original paper is practical and useful. But a normalization factor needs to be pointed out to make the table ready to use.**

Consider the numerical example in section 4 of [1], and see that paper for notation. Without applying the age replacement policy, the failure rate is 1/14 per hr and the cost-rate of operation is \$20/14 hours = \$1.43/hour. And yet, under the optimum age replacement, the $\$(T)$, cost-rate of operation was \$17.61/hour.

The reason for the apparent discrepancy is an implicit normalization. $\$(T)$ is normalized by the Weibull distribution scale parameter, α. Namely $\$(T)$ is actually the mean cost of operation per α units of time.

Now, rewrite (2) & (3) of [1] as follows:

$$\$(t)\alpha = [1 + (c-1)\,\text{weifc}(t/a;\beta)]\,c_1/[\text{ML}(t)/\alpha] \qquad (2)'$$

$$1 + (c-1)\,\text{weifc}(t/\alpha;\beta) + \beta(c-1)(t/\alpha)^{\beta-1}\,[\text{ML}(t)/\alpha] = 0 \qquad (3)'$$

By using this numerical example, eq. (3)′ can be checked to satisfy the new interpretation of $\$(t)$:

$$\alpha = 15.8, \quad \beta = 1.8, \quad T/\alpha = 0.541,$$

$$\text{weifc}(T/\alpha;\beta) = 0.7182$$

By (2)′, $\text{ML}(T)/\alpha = 0.4829$ when $\$(T)\alpha/c_1$ (rather than $\$(T)/c_1$) is taken as 0.881. Substituting these values into (3)′ yields:

$$0.4254 + 1.8\,(-0.8)(0.541)^{0.8}(0.4829) = 0.$$

This shows that $\$(T)$ as used in [1] is in fact α times the cost rate, eg, for the example —

$$\$(T) = \$17.61/\alpha = \$17.61/15.8 \text{ hours} = \$1.11/\text{hour}.$$

Author Reply

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I thank T. Y. Liang for calling this oversight to the attention of myself and the readers. Please note that the decision-maker chooses only T, and that calculation is correct.

REFERENCE

[1] P. R. Tadikamalla, "Age replacement policies for Weibull failure times," *IEEE Trans. Reliability*, vol R--29, 1980 Apr, pp 88-90.

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