Math 112
Calculus Lab 2

Applications of Integration

In previous work, we have studied several ways of evaluating integrals. With these tools in hand, it is possible to explore some of the many applications of the definite integral by solving problems in areas such as geometry and physics. There are many situations in which the quantity of interest may be expressed as a definite integral. The general procedure, which we followed when we found the area under curves, is to divide the quantity of interest into small pieces, solve the problem approximately for each small piece, and then sum the resulting approximations. For instance, in the area problem, the interval of interest was divided into n smaller subintervals. Summing the areas under the curve over each subinterval yielded a Riemann sum, and the definite integral came about as the limit of the Riemann sum as the subintervals became smaller and smaller. In many problems however, the difficulty lies in recognizing the quantity that we want as a Riemann sum. Therefore, the key in using the technique of "slicing and summing" is to properly divide the quantity of interest into small subdivisions, the sum of which lead to a Riemann sum. Then, taking the limit as the subdivisions get smaller and smaller leads to a definite integral.

Introduction

The basic goal is to calculate a variety of quantities by setting up and evaluating a definite integral. The general procedure may be outlined as follows:

Step 1: Chop up the desired quantity into very thin slices.

Step 2: Within each slice, calculate an approximation to the desired quantity.

Step 3: Add up the results of all of the slice approximations from Step 2. The resulting sum is a Riemann sum that approximates the desired total quantity.

Step 4: Obtain a definite integral by taking the limit of the Riemann sum in Step 3 as the slices get thinner and thinner.

Many problems in application will involve the idea of density. For instance in calculating the total population of a city, we might use a population density, or number of people per square mile. We might also run into the problem of calculating the total mass of some substance. In this case, we would use the density of the substance (*i.e.* air or water) given as a mass per unit volume in, say, grams per cubic centimeter. Whatever quantity is given, the key point to remember is that we want to slice the problem in such a way as to keep the quantity of interest nearly constant within each piece.

1. An Example

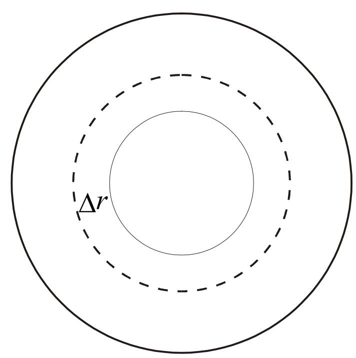
The density of oil in a circular oil slick on the surface of the ocean, at a distance *r* meters from the center of the slick, is given by

$$\rho(r) = 50/(1+r) \quad [\text{kg/m}^2].$$

The slick extends from $r = 0$ to $r = 10{,}000$ m.

a) In what direction should the slick be sliced in order to obtain slabs with nearly constant density? (Note: the density of the oil is a function of radial distance from the center of the slick).

b) If the slick is sliced into concentric rings, we are faced with the problem of approximating the mass of oil contained in rings of thickness Δr. Since the rings are very thin, we may approximate the area of each by thinking of them as rectangles with length $2\pi r$ (the circumference of the ring) and width Δr. (Think about unwrapping each ring and straightening it out into a long, thin rectangle.) By using this approximation for the area of each ring, write a definite integral that could be used to find the total mass of oil in the slick. (Note: the approximation for the area of each ring may also be obtained by subtracting the area of the inner circle $\pi(r - \Delta r/2)^2$ from the area of the outer circle $\pi(r + \Delta r/2)^2$.)



c) Find the exact value of the mass of the oil slick by evaluating the integral from part (b) using Maple.

2. Work

In physics, the term work has a meaning that differs from its everyday usage. Physicists say that if a constant force (*i.e.* a push or a pull) is applied to some object to move it some distance, d, then the force has done work on the object. This definition of work may be written as

$$\text{Work done} = \text{Force} \times \text{Distance}$$

$$W = F \cdot d \ .$$

Notice that according to this definition, if the object does not move, then no work is done. So, if we hold a book at the same height for some time, no work is done. A force must be exerted to keep the book from falling (to counteract the force of gravity), but the book is not moved through a distance. Additionally, if we walked across the room while holding the book at the constant height, we would not accomplish any work on the book, since the force that we exerted is in the vertical direction and the movement of the book was in the horizontal direction. In other words, the direction of motion must have a component in the direction of the applied force for work to be performed.

The definition provided above defines work as long as the force is constant. If we are faced with a problem in which the force is variable, we need to make some provisions. However, we already know a procedure for getting around this problem, namely slicing and summing. Faced with a variable force, we slice the problem in such a way that the force may be assumed constant over each piece. We then calculate the work for each slice using $W = Fd$, and then sum the pieces to approximate the total work.

We may make this point clear by looking at the following problem.

Example: Water is to be raised from a well 40 ft. deep by means of a bucket attached to a rope. When the bucket is full, it weighs 30 lbs.

a) Neglecting the weight of the rope, find the work required to lift a full bucket out of the well.

b) Now, suppose the bucket has a leak that causes it to lose water at the constant rate of 0.25 lb. for each foot the bucket is raised. If height is measured from the water level in the well, what is the weight of the bucket at x ft above the water?

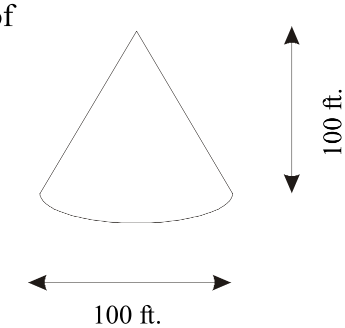
c) Using the result from part b, what is the work required to move the bucket a distance Δx?

d) Using the results from part (c) write the integral needed to find the total amount of work done in raising the leaking bucket to the top of the well. You may evaluate the integral by hand or using Maple.

3. Another Example

A monument in the shape of a cone is built to a height of 100 ft. The base has a diameter of 100 ft. and the bricks used in construction weigh 2 lbs/ft^3.

a) How should this monument be sliced to ultimately calculate the work required to build it - vertically or horizontally? Sketch your slicing scheme below. (Hint: work is being performed to counteract the force of gravity.)



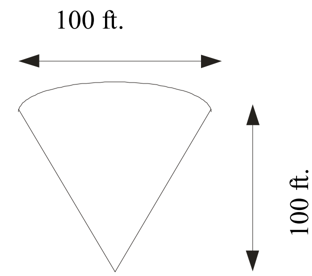
b) In finding the work required to construct the i^{th} slice of the monument, we need to know the weight of the bricks in that slice. We know that the bricks weigh 2 lb/ft^3, so multiplying by the volume of the slice will give the weight of the bricks in the slice. If height, h, is measured from the base of the monument, find the volume of the i^{th} slice of the monument. (Hint: Use similar triangles to find the radius of the slice as a function of h.)

c) Using the result of part b, find and evaluate a definite integral that represents the total work required to build the monument. You may show your work by hand or attach your Maple worksheet.

Homework for Lab 2 Math 112

Name__________________
Section number _________

1. Suppose that the same conical structure from problem 4 is to be built in another location, except that it will be inverted.



a) Neglecting any additional structure to keep the monument inverted, write a definite integral that represents the total work required to build the monument.

b) Label any variables you use on the diagram.

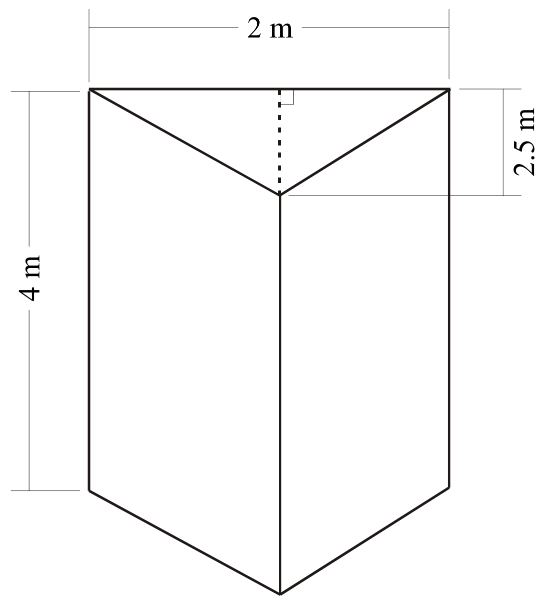
c) Use Maple to evaluate the integral. Attach your Maple worksheet to this page.

d) Considering the definition of work given previously, provide an intuitive explanation of the results of problem 4 in the lab and this problem. (In other words, explain why it takes more work to build the monument upside down.)

2. A certain compressible liquid has a density, ρ, which varies with height, h, above the bottom of its container is given by

$$\rho(h) = 40(5-h)\ \frac{\text{kg}}{\text{m}^3}.$$

a) The liquid is placed in the container shown on the right. The cross sections of the container are isosceles triangles. It has straight sides and looks like a triangular prism. How many kg will it hold when placed as shown, resting on the triangular side?
Write the integral needed to solve the problem and evaluate it by hand or using Maple.



b) How many kg will the same container as part a hold if it is placed (with some support, of course) as shown below?

Write the integral needed to solve the problem and evaluate it by hand or using Maple.