Existence of a common solution to the Lyapunov equation
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A matrix $A \in \mathbb{C}^{n \times n}$ is called (Hurwitz) stable if all its eigenvalues lie in the open left half of the complex plane. A classical result of Lyapunov states that a matrix $A$ is stable if and only if for arbitrary Hermitian positive definite $Q$, the Lyapunov equation

$$AP + PA^* = -Q$$

admits a positive definite solution $P$. The problem of determining when for given stable matrices $A_1, A_2, \ldots, A_k$ in $\mathbb{C}^{n \times n}$ there exists a common solution $P > 0$ to the Lyapunov equations:

$$A_j P + PA_j^* < 0$$

for $j = 1, 2, \ldots, k$, is important in applied and theoretical research.

We will present some conditions for two stable complex matrices to have a common solution to the Lyapunov equation. We will apply these conditions to the case when a common weak solution exists and show that necessary and sufficient conditions for the existence of a common solution for $2 \times 2$ complex matrices $A$ and $B$ is that matrices $(A + i\alpha I)(B + i\beta I)$ and $(A + i\alpha I)^{-1}(B + i\beta I)$ have no negative real eigenvalues for all real numbers $\alpha$ and $\beta$. 

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