Existence of a common solution to the Lyapunov equation

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A matrix $A \in \mathbb{C}^{n\times n}$ is called (Hurwitz) stable if all its eigenvalues lie in the open left half of the complex plane. A classical result of Lyapunov states that a matrix A is stable if and only if for arbitrary Hermitian positive definite Q, the Lyapunov equation

$$AP + PA^* = -Q$$

admits a positive definite solution P. The problem of determining when for given stable matrices $A_1, A_2, \ldots, A_k$ in $\mathbb{C}^{n\times n}$ there exists a common solution $P > 0$ to the Lyapunov equations:

$$A_j P + PA_j^* < 0$$

for $j = 1, 2, \ldots, k$, is important in applied and theoretical research.

We will present some conditions for two stable complex matrices to have a common solution to the Lyapunov equation. We will apply these conditions to the case when a common weak solution exists and show that necessary and sufficient conditions for the existence of a common solution for 2×2 complex matrices A and B is that matrices $(A + \mathrm{i}\alpha I)(B + \mathrm{i}\beta I)$ and $(A + \mathrm{i}\alpha I)^{-1}(B + \mathrm{i}\beta I)$ have no negative real eigenvalues for all real numbers α and β.