Locally 3-arc-transitive regular covers of complete bipartite graphs

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Abstract. In this paper, locally 3-arc-transitive regular covers of complete bipartite graphs are studied, and results are obtained that apply to arbitrary covering transformation groups. In particular, methods are obtained for classifying the locally 3-arc-transitive graphs with a prescribed covering transformation group, and these results are applied to classify the locally 3-arc-transitive regular covers of complete bipartite graphs with covering transformation group isomorphic to a cyclic group or an elementary abelian group of order $p^2$.

1. Introduction

We will assume that all graphs are finite, simple, undirected, and connected, unless otherwise stated. An $s$-arc of a graph $\Gamma$ is a sequence of vertices $(\alpha_0, \alpha_1, ..., \alpha_s)$ such that $\alpha_i$ is adjacent to $\alpha_{i+1}$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for all $i$, $1 \leq i \leq s - 1$. Note that vertices can be repeated as long as $\alpha_{i-1} \neq \alpha_{i+1}$. An automorphism of the graph $\Gamma$ is a permutation of the vertices that preserves adjacency and nonadjacency. The set of all automorphisms of the graph $\Gamma$ forms a group and is denoted $\text{Aut}(\Gamma)$. Given a subgroup $G \leq \text{Aut}(\Gamma)$, $\Gamma$ is $(G, s)$-arc-transitive if $\Gamma$ contains an $s$-arc and any $s$-arc in $\Gamma$ can be mapped to any other $s$-arc in $\Gamma$ via an element of $G$. The graph is locally $(G, s)$-arc-transitive if $\Gamma$ contains an $s$-arc and, for any vertex $\alpha$ of $\Gamma$, any $s$-arc starting at $\alpha$ can be mapped to any other $s$-arc starting at $\alpha$ via an element of $G$. In the cases that such a group $G$ exists, the graph $\Gamma$ is said to be $s$-arc-transitive or locally $s$-arc-transitive, respectively. Note that it is possible for a graph to be locally $(G, s)$-arc-transitive but for $G$ to be intransitive on the set of vertices. (As an example, one could take the complete bipartite graph $K_{2,3}$.) On the other hand, when $\Gamma$ is locally $(G, s)$-arc-transitive and every vertex in $\Gamma$ is adjacent to at least two other vertices, $G$ is transitive on the edges of $\Gamma$.

The study of $s$-arc-transitive graphs began with Tutte’s seminal studies of cubic graphs [27, 28]. Tutte proved that if $\Gamma$ is a cubic $(G, s)$-arc-transitive graph, then $s \leq 5$ (and that this upper bound is the best possible). Later, Weiss demonstrated that if the valency is at least three, then $s \leq 7$ [30]. Whereas Tutte’s result used clever elementary methods, Weiss’s result relied on the Classification of Finite Simple Groups (CFSG). On the other hand, locally $s$-arc-transitive graphs give geometric descriptions of rank 2 amalgams. The proof that $s \leq 9$ for any locally $s$-arc-transitive graph with all vertex valencies at least three [29] relies heavily on knowledge of amalgams and the classification of weak $(B,N)$-pairs of rank 2 by Delgado and Stellmacher [2] and further demonstrates the deep link between the two concepts. Moreover, locally $s$-arc-transitive graphs arise as incidence graphs of highly symmetric structures [19, 31] and are important in their study. For

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instance, the only examples of Moufang generalized octagons are the Ree-Tits generalized octagons \[26\], the incidence graphs of which are locally 9-arc-transitive graphs.

A recent paper by Giudici, Li, and Praeger \[11\] set forth a program for the study of locally \(s\)-arc-transitive graphs. First, any locally \((G, s)\)-arc-transitive graph that is \(G\)-vertex transitive is a \((G, s)\)-arc-transitive graph, and many have been classified under a similar program initiated by Praeger in \[23\]. Second, since any locally \((G, s)\)-arc-transitive graph of valency at least two is edge transitive, any locally \((G, s)\)-arc-transitive graph that is vertex intransitive must be bipartite with vertex set \(VT = \Delta_1 \cup \Delta_2\) such that \(G\) is transitive on each of \(\Delta_1\) and \(\Delta_2\), respectively. Third, there is a characterization of the graphs containing vertices of valency two \[12\, Theorem 6.2, Corollary 6.3\]. Thus, the focus is on the graphs with minimum valency at least three that are \(G\)-vertex intransitive.

The breakthrough for studying these remaining \(G\)-vertex intransitive graphs is the normal quotient method from \[11\]. Let \(\Gamma\) be a graph with a group of automorphisms \(G\). If \(G\) has a normal subgroup \(N\) that acts intransitively on the vertices of \(\Gamma\), define the (normal) quotient graph \(\Gamma_N\) to have vertex set the \(N\)-orbits of vertices of \(\Gamma\), and two \(N\)-orbits \(\Sigma_1\) and \(\Sigma_2\) are adjacent in \(\Gamma_N\) if and only if there exist vertices \(\alpha \in \Sigma_1\) and \(\beta \in \Sigma_2\) such that \(\alpha\) is adjacent to \(\beta\) in \(\Gamma\). The original graph \(\Gamma\) is said to be a regular cover of \(\Gamma_N\) if each \(N\)-orbit \(\Sigma_2\) adjacent to \(\Sigma_1\) contains exactly one vertex adjacent to \(\alpha\) in \(\Gamma\) for each \(\alpha \in \Sigma_1\), and in this case \(N\) is referred to as the covering transformation group. (See also Section 2 for different yet equivalent topological definitions.) Giudici, Li, and Praeger \[11\] showed that if \(\Gamma\) is a locally \((G, s)\)-arc-transitive graph, then \(\Gamma_N\) is a locally \((G/N, s)\)-arc-transitive graph. This insight led to the following framework for studying locally \(s\)-arc-transitive graphs:

(I) Understand the basic graphs, that is, those for which there is no nondegenerate normal quotient. A single vertex, a single edge, and a star, i.e., graphs of the form \(K_{1,m}\) for some \(m > 1\) \[11\, Theorem 1.1\], are the degenerate normal quotients in this context.

(II) Understand the regular covers of the basic graphs.

The basic graphs will arise when the group \(G\) acting on the graph has no normal subgroup with more than two orbits. With this in mind, a group \(G\) acting transitively on a set \(\Omega\) is said to be quasiprimitive if every nontrivial normal subgroup \(N\) of \(G\) is transitive on \(\Omega\). Given the above definition of a quotient graph, it is clear that quasiprimitive groups are precisely the groups that give rise to basic graphs. The quasiprimitive groups were classified by Praeger \[23\] and refined into the following eight types \[24\]: holomorph of abelian group (HA), holomorph simple (HS), compound holomorph (HC), almost simple (AS), twisted wreath product (TW), simple diagonal (SD), compound diagonal (CD), and product action (PA). The main distinguishing factor among these different quasiprimitive actions is the socle of the quasiprimitive group \(G\), which is the subgroup \(soc(G)\) of \(G\) generated by all minimal normal subgroups of \(G\). There are four fundamentally different kinds of basic graphs for vertex intransitive locally \((G, s)\)-arc-transitive graphs, where \(s \geq 2\):

[Complete Bipartite]: Complete bipartite graphs \(K_{m,n}\), where \(m \neq n\);

[Star Normal Quotient]: \(G\) acts faithfully on both orbits of vertices \(\Delta_1\) and \(\Delta_2\), but only acts quasiprimittively on \(\Delta_1\). In this case, the quasiprimitive action of \(G\) on \(\Delta_1\) must be one of HA, HS, AS, PA, or TW \[11\, Theorem 1.3\];

[Same Type]: \(G\) acts faithfully and quasiprimittively on both orbits of vertices \(\Delta_1\) and \(\Delta_2\) with the same quasiprimitive type on each part, and the action of \(G\) must be one of HA, TW, AS, or PA \[11\, Theorem 1.2\].
**Theorem 1.1.** Let $\Gamma = K_{m,n}$ with biparts of vertices $\Delta_1$ and $\Delta_2$, $|\Delta_1| = m \leq n = |\Delta_2|$, $\mathcal{P} : \hat{\Gamma} \to \Gamma$ be a regular covering projection such that $H := \text{CT}(\mathcal{P})$ and $G \leq \text{Aut}(\Gamma)$ lifts to $\tilde{G}$ such that $\tilde{\Gamma}$ is locally $(\tilde{G}, 3)$-arc-transitive. For vertices $\alpha, \beta \in \Delta_\gamma$, define further $K_{m,n}^{\alpha,\beta} \cong K_{2,n}$ to be the subgraph induced on $\{\alpha, \beta\} \cup \Delta_i$ and $H_\beta$ to be the covering transformation group of $\mathcal{P}_{\alpha,\beta} : \tilde{\Sigma} \to K_{m,n}^{\alpha,\beta}$, where $\tilde{\Sigma}$ is a connected component of $\mathcal{P}^{-1}(K_{m,n}^{\alpha,\beta})$. Then one of the following must hold for each $\Delta_i$:

1. $|\Delta_i| = p^f$ for some prime $p$ and $f \in \mathbb{N}$, and $H_{\beta}$ is an elementary abelian group of order $p^f$, or

2. There exists $X \leq \text{Aut}(H_{\beta})$ such that $X$ has a 2-transitive action on $|\Delta_i|$ points. Moreover, in either case, if $v \in \Delta_i$, then $G_{\alpha,\beta}^{\Delta_i}$ is isomorphic to a subgroup of $\text{Aut}(H_{\beta})$.

One inference that we can make from Theorem 1.1 is that elementary abelian covering transformation groups are the most likely to yield locally 3-arc-transitive regular covers of complete bipartite graphs. We apply Theorem 1.1 to cyclic covering transformation groups and elementary abelian covering transformation groups in Sections 3 and 4 respectively, and obtain the following classification results:

**Theorem 1.2.** Let $\Gamma = K_{m,n}$, $m \leq n$, $\mathcal{P} : \hat{\Gamma} \to \Gamma$ be a regular covering projection such that $H := \text{CT}(\mathcal{P}) \neq 1$ is a cyclic group and $G \leq \text{Aut}(\Gamma)$ lifts to $\tilde{G}$ such that $\tilde{\Gamma}$ is locally $(\tilde{G}, 3)$-arc-transitive. Then one of the following holds:

1. $m = n = 2$ and $H \cong \mathbb{Z}_N$, where $N$ is any positive integer at least 2;
2. $m = 2, n = p$ for some odd prime $p$, and $H \cong \mathbb{Z}_p$;
3. $m = n = p$ for some odd prime $p$ and $H \cong \mathbb{Z}_p$. 

While progress has been made in understanding the basic graphs [9] [10] [11] [12] [13], [19] [25], there has been very little work done toward understanding the regular covers of complete bipartite graphs. In Section 2, we obtain results that apply to locally 3-arc-transitive regular covers of complete bipartite graphs. We apply Theorem 1.1 to cyclic covering transformation groups, culminating in the following result:

In order to introduce the main results of this paper, we need the following notation.

Let $\alpha$ be a vertex of a graph $\Gamma$, and let $G \leq \text{Aut}(\Gamma)$. We denote the vertices adjacent to $\alpha$ in $\Gamma$ by $\Gamma(\alpha)$, and the permutation group induced by $G_\alpha$ on $\Gamma(\alpha)$ will be denoted by $G_\alpha^{\Gamma(\alpha)}$. For any two vertices $\alpha, \beta$ of $\Gamma$, we define the distance function $d(\alpha, \beta)$ to be the length of the shortest path between $\alpha$ and $\beta$ in $\Gamma$. Given a natural number $i$, we define

$$G_\alpha^{[i]} := \{g \in G \mid \beta^g = \beta \text{ for all } \beta \text{ satisfying } d(\alpha, \beta) \leq i\}.$$ 

Note that $G_\alpha^{[1]}$ is a normal subgroup of $G_\alpha$, and $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha / G_\alpha^{[1]}$.

In this paper, we study the locally 3-arc-transitive regular covers of complete bipartite graphs. In Section 2 we obtain results that apply to locally 3-arc-transitive regular covers of complete bipartite graphs with arbitrary covering transformation group, culminating in the following result:

**Theorem 1.1.** Let $\Gamma = K_{m,n}$ with biparts of vertices $\Delta_1$ and $\Delta_2$, $|\Delta_1| = m \leq n = |\Delta_2|$, $\mathcal{P} : \hat{\Gamma} \to \Gamma$ be a regular covering projection such that $H := \text{CT}(\mathcal{P})$ and $G \leq \text{Aut}(\Gamma)$ lifts to $\tilde{G}$ such that $\tilde{\Gamma}$ is locally $(\tilde{G}, 3)$-arc-transitive. For vertices $\alpha, \beta \in \Delta_\gamma$, define further $K_{m,n}^{\alpha,\beta} \cong K_{2,n}$ to be the subgraph induced on $\{\alpha, \beta\} \cup \Delta_i$ and $H_\beta$ to be the covering transformation group of $\mathcal{P}_{\alpha,\beta} : \tilde{\Sigma} \to K_{m,n}^{\alpha,\beta}$, where $\tilde{\Sigma}$ is a connected component of $\mathcal{P}^{-1}(K_{m,n}^{\alpha,\beta})$. Then one of the following must hold for each $\Delta_i$:

1. $|\Delta_i| = p^f$ for some prime $p$ and $f \in \mathbb{N}$, and $H_{\beta}$ is an elementary abelian group of order $p^f$, or

2. There exists $X \leq \text{Aut}(H_{\beta})$ such that $X$ has a 2-transitive action on $|\Delta_i|$ points. Moreover, in either case, if $v \in \Delta_i$, then $G_{\alpha,\beta}^{\Delta_i}$ is isomorphic to a subgroup of $\text{Aut}(H_{\beta})$.

One inference that we can make from Theorem 1.1 is that elementary abelian covering transformation groups are the most likely to yield locally 3-arc-transitive regular covers of complete bipartite graphs. We apply Theorem 1.1 to cyclic covering transformation groups and elementary abelian covering transformation groups in Sections 3 and 4 respectively, and obtain the following classification results:

**Theorem 1.2.** Let $\Gamma = K_{m,n}$, $m \leq n$, $\mathcal{P} : \hat{\Gamma} \to \Gamma$ be a regular covering projection such that $H := \text{CT}(\mathcal{P}) \neq 1$ is a cyclic group and $G \leq \text{Aut}(\Gamma)$ lifts to $\tilde{G}$ such that $\tilde{\Gamma}$ is locally $(\tilde{G}, 3)$-arc-transitive. Then one of the following holds:

1. $m = n = 2$ and $H \cong \mathbb{Z}_N$, where $N$ is any positive integer at least 2;
2. $m = 2, n = p$ for some odd prime $p$, and $H \cong \mathbb{Z}_p$;
3. $m = n = p$ for some odd prime $p$ and $H \cong \mathbb{Z}_p$. 

The graphs in [Star Normal Quotient] have a (degenerate) quotient that is isomorphic to the star $K_{1,m}$ for some $m > 1$. This is the reason these graphs are collectively referred to as those with a star normal quotient.

While progress has been made in understanding the basic graphs [9] [10] [11] [12] [13], [19] [25], there has been very little work done toward understanding the regular covers of large families of symmetric graphs collectively has only been undertaken in a few specific cases [6] [7] [16], even though studying regular covers of large families of symmetric graphs is necessary to understanding symmetric graphs as a whole.

The graphs in [Star Normal Quotient] have a (degenerate) quotient that is isomorphic to the star $K_{1,m}$ for some $m > 1$. This is the reason these graphs are collectively referred to as those with a star normal quotient.
Moreover, in each case the cover is unique (up to isomorphism of covering projections).

**Theorem 1.3.** Let \( \Gamma = K_{m,n} \) with biparts of vertices \( \Delta_1 \) and \( \Delta_2 \), \( |\Delta_1| = m \leq n = |\Delta_2| \), \( \mathcal{P} : \tilde{\Gamma} \to \Gamma \) be a regular covering projection such that \( H := \text{CT}(\mathcal{P}) \cong \mathbb{Z}_d^p \), \( p \) prime and \( d \geq 1 \), is an elementary abelian group and \( G \leq \text{Aut}(\Gamma) \) lifts to \( \tilde{G} \) such that \( \tilde{\Gamma} \) is locally \((\tilde{G}, 3)\)-arc-transitive. Then one of the following must hold for each \( \Delta_i \):

1. \( |\Delta_i| = p^f \) for some \( f \leq d \), or
2. There exists \( X \leq \text{GL}_d(p) \) such that \( X \) has a 2-transitive action on \( |\Delta_i| \) points.

Moreover, in either case, if \( v \in \Delta_i \), then \( G^{\Delta_i}_v \) is isomorphic to a subgroup of \( \text{GL}_d(p) \).

We apply this result to covering transformation groups isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \) to obtain the following classification:

**Theorem 1.4.** Let \( H \cong \mathbb{Z}_p \times \mathbb{Z}_p \), where \( p \) is a prime. The only complete bipartite graphs with a locally 3-arc-transitive regular cover whose covering transformation group is isomorphic to a subgroup of \( H \) are \( K_{2,2}, K_{2,3}, K_{2,p}, K_{2,p^2}, K_{3,p}, K_{3,p^2}, K_{p,p}, K_{p,p^2} \), and \( K_{p^2,p^2} \).

**Corollary 1.5.** Let \( \Gamma = K_{m,n} \), \( m \leq n \), \( \mathcal{P} : \tilde{\Gamma} \to \Gamma \) be a regular covering projection such that \( H := \text{CT}(\mathcal{P}) \cong \mathbb{Z}_p \times \mathbb{Z}_p \) is an elementary abelian group of order \( p^2 \) and \( G \leq \text{Aut}(\Gamma) \) lifts to \( \tilde{G} \) such that \( \tilde{\Gamma} \) is locally \((\tilde{G}, 3)\)-arc-transitive. Then one of the following holds:

1. \( m = 2 \) and \( n = 3 \);
2. \( m = 2 \) and \( n = p^2 \);
3. \( m = 3 \) and \( n = p \);
4. \( m = 3 \) and \( n = p^2 \);
5. \( m = p \) and \( n = p^2 \);
6. \( m = n = p^2 \).

Moreover, in each case the cover is unique (up to isomorphism of covering projections).

2. Results applicable to general covering transformation groups

A covering projection \( \mathcal{P} : \tilde{\Gamma} \to \Gamma \) maps \( V\tilde{\Gamma} \) onto \( V\Gamma \), preserving adjacency, such that for any vertex \( \tilde{\alpha} \in V\tilde{\Gamma} \), the set of neighbors of \( \tilde{\alpha} \) is mapped bijectively onto the neighbors of \( \mathcal{P}(\tilde{\alpha}) \). For a vertex \( \alpha \) of \( \Gamma \), the set of vertices \( \mathcal{P}^{-1}(\alpha) \) that are mapped onto \( \alpha \) by \( \mathcal{P} \) is called the fiber over the vertex \( \alpha \). An automorphism \( g \in \text{Aut}(\Gamma) \) lifts to \( \tilde{g} \in \text{Aut}(\tilde{\Gamma}) \) if the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{\Gamma} & \xrightarrow{\tilde{g}} & \tilde{\Gamma} \\
\mathcal{P} \downarrow & & \mathcal{P} \downarrow \\
\Gamma & \xrightarrow{g} & \Gamma 
\end{array}
\]

The lift of the trivial group (identity) is known as the group of covering transformations and is denoted \( \text{CT}(\mathcal{P}) \). \( \tilde{\Gamma} \) is a regular cover of \( \Gamma \) if \( \text{CT}(\mathcal{P}) \) acts regularly on the set \( \mathcal{P}^{-1}(\alpha) \) for all vertices \( \alpha \in V\Gamma \). (Compare these to the equivalent definitions of Section 3.)

A voltage assignment is a map \( \xi : \tilde{E}\Gamma \to H \), where \( H \) is a group, such that \( \xi(\alpha, \beta) = \xi(\beta, \alpha)^{-1} \). For ease of notation, the voltage of the arc \((\alpha, \beta)\) will be denoted \( \xi_{\alpha\beta} \). The derived covering graph \( \tilde{\Gamma} \) of a voltage graph has vertex set \( V\Gamma \times H \), where two vertices
(α, h₁) and (β, h₂) are adjacent iff α is adjacent to β in Γ and h₂ = ξ_{αβ}h₁. The following theorem exhibits the deep connection between regular covers and derived covering graphs:

**Theorem 2.1 ([15] Theorem 2.4.5, Section 2.5)**. Every regular cover \( \hat{\Gamma} \) of a graph Γ is a derived cover of a voltage graph. Moreover, if \( \hat{\Gamma} \) is connected we may assume that the edges of a (fixed but arbitrary) spanning tree of Γ have been assigned the identity voltage.

Two covering projections \( \mathcal{P}_1 : \hat{\Gamma}_1 \to \Gamma_1 \) and \( \mathcal{P}_2 : \hat{\Gamma}_2 \to \Gamma_2 \) are said to be isomorphic if there exist graph isomorphisms \( \sigma : \Gamma_1 \to \Gamma_2 \) and \( \tilde{\sigma} : \hat{\Gamma}_1 \to \hat{\Gamma}_2 \) such that \( \sigma \mathcal{P}_1 = \mathcal{P}_2 \tilde{\sigma} \). The following lemma connects isomorphisms of covers to isomorphisms of the covering transformation groups:

**Lemma 2.2 ([21] Corollary 3.3)**. Let \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) be two regular covers of a connected graph Γ, and let \( H_1 = \text{CT}(\mathcal{P}_1) \) and \( H_2 = \text{CT}(\mathcal{P}_2) \). The covering projections \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are isomorphic if and only if there exists a group isomorphism \( \tau : H_1 \to H_2 \) and \( \sigma \in \text{Aut}(\Gamma) \) such that \( \tau \mathcal{P}_1 = \mathcal{P}_2 \sigma \). The following lemma gives explicit criteria for an automorphism of a graph to lift:

**Lemma 2.3 ([21] Propositions 3.1, 5.1)**. An automorphism \( g \) of Γ lifts to an automorphism \( \hat{g} \) of \( \hat{\Gamma} \) if and only if there exists a group automorphism \( g^{#α} \in \text{Aut}(H) \) such that \( \xi_W^{g^{#α}} = \xi_W \), where \( W \) runs over a generating set for all closed walks based at a single vertex \( α \). Moreover, if \( H \) is abelian, the automorphism \( g^{#α} \) does not depend on the choice of base vertex \( α \).

Note that by Giudici, Li, Praeger [11] Theorem 1.1, if \( G = \hat{G}/\text{CT}(\mathcal{P}) \), then \( Γ \) is a locally (G, 3)-arc-transitive graph, and the group of automorphisms \( G \) of Γ lifts to \( \hat{\Gamma} \).

At this point, we may associate each voltage \( \xi_i \) with the closed walk \( (α, i, γ, 0, α) \) based at \( α \). We note that for any fixed \( β \neq α \in Δ_1 \) and any \( g \in G_{αβ} \), we may use Lemma 2.3 to define a homomorphism \( φ_{αβ} : G_{αβ} \to \text{Aut}(H) \) as follows: for any \( g \in G_{αβ} \), we choose the lift \( \hat{g} \) of \( g \) such that \( (α,1)β = (αγ,1) \). For a given \( h \in H \), we may define the induced automorphism on \( H \) by the action of \( \hat{g} \) on \( H \) via the mapping \( (α, h)β = (αγ, hφ_{αβ}(g)) \). Furthermore, we note that \( \xi_{iγ}^{φ_{αβ}(g)} = ξ_{iγ}^{-1} ξ_{iφ_{γα}(0)} ξ_{iγ}^{φ_{αβ}(g)} \), where \( ξ_{iφ_{αβ}(0)} = 1 \) for any \( δ \in Δ_1 \).

**Lemma 2.4.** For any vertex \( β \neq α \in Δ_1 \), \( G_{αβ}^{Δ_2} \) is 2-transitive. Moreover, the voltages assigned to the co-tree edges of Γ must all have the same order, and, for any two distinct vertices \( i, j \in Δ_2 \), \( ξ_{ij} \neq ξ_{j} \).

**Proof.** The fact that \( G_{αβ}^{Δ_2} \) is 2-transitive follows from local 3-arc-transitivity. Note that any 3-arc beginning at \( α \) and ending at \( 0 \), say \( (α, i, β, 0) \), defines a unique 4-cycle...
(α, i, β, j, γ). Since Γ is locally (G, 3)-arc-transitive, for any γ ∈ Δ1 and j ∈ Δ2, there exists g ∈ G such that (α, i, β, j, γ) = (α, j, β, γ, 0). Hence the voltage of the first walk, ξiβ, must be mapped to the voltage of the second walk, ξjγ, by the induced action of the element g on H (i.e., ϕαβ(g) ∈ Aut(H), as defined above). Thus the voltages on co-tree edges have the same order. Finally, if ξiβ = ξjβ for some i ≠ j, then the voltage of the closed walk (α, i, β, j, γ) is trivial. However, for any 0 ≠ k ∈ Δ2 and α ≠ γ ∈ Δ1, there is g ∈ Ga such that (α, i, β, j, γ) = (α, k, γ, 0), and so ξkγ = 1 ∈ Gαβ(g) = 1, and all co-tree edges have trivial voltage.

Recall that the socle of a group X, denoted by soc(X) is defined to be the subgroup generated by all minimal normal subgroups of X. Since Gαβ is a 2-transitive group for any β ≠ α ∈ Δ1, by Burnside’s Theorem (see [14] Theorem 4.1B), for instance, soc(Gαβ) is either an elementary abelian p-group that acts regularly on Δ2 for some prime p or a nonabelian simple group that acts nonregularly on Δ2. Moreover, in each case the socle is the unique minimal normal subgroup. Henceforth we will assume that β ≠ α ∈ Δ1 is fixed and let S := soc(Gαβ).

We say that a group X is involved in the group Y if there exist subgroups Y1 ⊆ Y2 ≤ Y such that Y2/Y1 ≅ X. Let φ : Gαβ → Gαβ be the natural projection with Ker(φ) = Gα3. Since Gαβ has a unique minimal normal subgroup S, by the Correspondence Theorem there exists a subgroup T ⊆ Gαβ such that Gα3 ≤ T and T/Gα3 ≅ S. Note that, since S is a minimal normal subgroup of Gαβ, if N < T and N is normal in Gαβ, then N ≤ Gα3.

**Proposition 2.5.** Let P : ˜Γ → Γ be a covering projection with covering transformation group H. If ˜G is the lift of the group G, the cover ˜Γ is locally (G, 3)-arc-transitive, and S := soc(Gαβ) is a nonabelian finite simple group, then S is involved in Aut(H).

**Proof.** Let |Δ2| = n. First, we note that the group Gα30, the stabilizer of the vertex 0 ∈ Δ2 in the group Gαβ, transitively permutes the vertices 1, 2, ..., n − 1. Since ξiβ is the voltage of the cycle (α, i, β, 0, α), for any g ∈ Gα30, we have ξiβ = ξβiβ. By Lemma 2.4, ξiβ = ξβiβ only if i = j. Hence, for any such g ∈ Gα30, if it induces a trivial action on H, then iβ = i for all i, 1 ≤ i ≤ n − 1, which implies that g ∈ Gα3.

As noted above, there is T ≤ Gαβ such that Gα3 ≤ T and T/Gα3 ≅ S. If S is not involved in Aut(H), then T ≤ Ker(φαβ). However, by the Classification of Finite Simple Groups [14], the induced action of S0 on {1, ..., n − 1} is nontrivial. Hence T0 acts nontrivially on {1, ..., n − 1} and T is not contained in Ker(φαβ). This means that Ker(φαβ) ∩ T ≤ Gα3. Therefore, φαβ(T)/φαβ(Gα3) ≅ S, and S is involved in Aut(H).

**Proposition 2.6.** Let P : ˜Γ → Γ be a covering projection with covering transformation group H. If ˜G is the lift of the group G, the cover ˜Γ is locally (G, 3)-arc-transitive, and S := soc(Gαβ) is an elementary abelian p-group for some prime p, then S is involved in Aut(H) or each voltage assigned to a co-tree edge of Γ has order p.

**Proof.** Assume that S is not involved in Aut(H). As noted above, there is T ≤ Gαβ such that Gα3 ≤ T and T/Gα3 ≅ S. By the same argument as in the proof of Proposition 2.5, if g ∈ Gα30 and φαβ(g) = 1, then g ∈ Gα30. Hence, if S is not involved in Aut(H), T ≤ Ker(φαβ), and T0 = T ∩ Gα30 ≤ Gα3.

We note that T is transitive on Δ2, so there is s ∈ T such that 0s = 1. Recalling the epimorphism φ : Gαβ → Gαβ, φ(s) ∈ S and has order p. Thus sp ∈ Ker(φ) = Gα3, and φ(sn) = 0 for 1 ≤ m, n ≤ p − 1 implies m = n. This means that there exist
vertices $a_2, \ldots, a_{p-1} \in \Delta_2$ such that $1^s = a_2$, $a_m = a_{m+1}$ for $2 \leq m \leq p - 2$, and $a_{p-1}^* = 0$. On the one hand, $(\alpha, 1, \beta, 0, \alpha)^* = (\alpha, a_2, \beta, 1, \alpha)$, and so we have an induced action $\xi_{1_3}^{(\gamma \alpha)} = \xi_{1_3}^{-1} \xi_{a_2}$. On the other hand, we are assuming that $T \leq \text{Ker}(\varphi_{\alpha \beta})$, and so $\xi_{1_3}^{(\gamma \alpha)} = \xi_{1_3}$. This implies that $\xi_{1_3} = \xi_{1_3}^{(\gamma \alpha)} = \xi_{1_3}^{-1} \xi_{a_2}$, and so $\xi_{a_2} = \xi_{1_3}^2$. Proceeding similarly, by induction we see that $\xi_{\alpha \beta} = \xi_{1_3}^m$ for $1 \leq m \leq p - 1$.

Moreover, $G_{\alpha \beta}$ is 2-transitive on $\Delta_2$, so there exists $h \in G_{\alpha \beta}$ such that $0^h = 1$ and $1^h = 0$, which implies that $h s^{-1} = r \in G_{\alpha 0\beta}$. Noting that $0^r = 0$ and $1^r = a_{p-1}$, we have an induced action $\xi_{1_3}^{(\gamma \alpha)} = \xi_{1_3}^{-1}$. On the other hand,

$$\xi_{1_3}^{(\gamma \alpha)} = \left(\xi_{1_3}^{(\gamma \alpha)}(r) \right)^{\varphi_{\alpha \beta}(r)} = \xi_{a_{p-1}}^{(\gamma \alpha)} = \xi_{2_1 \gamma}^{(\gamma \alpha)}.$$

Hence $\xi_{1_3}^{-1} = \xi_{a_{p-1}}^{-1} = \xi_{1_3}$ and $\xi_{1_3}^p = 1$. Since $H \neq 1$, $|\xi_{1_3}| = p$, and by Lemma 2.4 the order of every voltage assigned to a co-tree edge of $\Gamma$ has order $p$.

**Proposition 2.7.** If the group $T$ has a trivial induced action on $H$, where $\phi(T) = S = \text{soc}(G_{\alpha \beta})$ and $G_{\alpha \beta}^{[1]} \leq T$, then each element of an entire elementary abelian group of order $p^f$, where $p^f$ is the size of the bipart $\Delta_2$, must be assigned as voltages to the edges incident with the vertex $\beta \neq \alpha \in \Delta_1$.

**Proof.** By Propositions 2.5 and 2.6, if the lift of $T$ acts trivially on $H$, then $G_{\alpha \beta}$ is an elementary abelian $p$-group that acts regularly on $\Delta_2$ for some prime $p$. Thus $|\Delta_2| = p^f$ for some $f \in \mathbb{N}$, and by Proposition 2.6, the order of each voltage assigned to a co-tree edge is $p$. For any $i \in \Delta_2 \setminus \{0\}$, there exists $s_i \in T$ such that $0^{s_i} = i$, and, proceeding as in the proof of Theorem 2.6, $\xi_{i_3}^m = \xi_{i_3}^{s_i^m}$, and so each power of the voltage $\xi_{i_3}$ is assigned to an edge incident with $\beta$.

We will next show that $\xi_{i_3}^{(\gamma \alpha)}(g)$ is the voltage assigned to some edge incident with $\beta$, where $i, j$ are distinct vertices in $\Delta_2 \setminus \{0\}$. In particular, $\xi_{i_3}^{-1} = \xi_{i_3}^{s_i} = \xi_{i_3}$ for some $i \in \Delta_2$. Since $G_{\alpha \beta}$ is 2-transitive on $\Delta_2$, there is $g \in G_{\alpha \beta}$ such that $(\alpha, i, \beta, 0, \alpha)^g = (\alpha, j, \beta, \ell, \alpha)$, and so $\xi_{i_3}^{(\gamma \alpha)}(g) = \xi_{i_3}^{-1} \xi_{i_3} = \xi_{i_3} \xi_{j_3}$. There is also an element $s \in T$ such that $0^s = \ell$, and so $gs^{-1} \in G_{\alpha 0\beta}$. Hence

$$\xi_{i_3} \xi_{j_3} = \xi_{i_3}^{(\gamma \alpha)}(g) = \left(\xi_{i_3}^{(\gamma \alpha)}(gs^{-1}) \right)^{\varphi_{\alpha \beta}(s)} = \xi_{i_3}^{(\gamma \alpha)}(s) = \xi_{i_3}^{(\gamma \alpha)}(s).$$

since by assumption $T$ has a trivial induced action on $H$. Hence all the voltages incident with $\beta$ are all distinct, have order $p$ (other than $\xi_{i_3}$, which is assigned the trivial voltage), and form a subgroup.

Finally, define $H_\beta$ to be the group of voltages that are assigned to edges incident with $\beta$. Note that the induced action of $G_{\alpha 0\beta}$ is transitive on $\Delta_2 \setminus \{0\}$, and hence for any $i \neq j \in \Delta_2 \setminus \{0\}$, there is $g \in G_{\alpha 0\beta}$ such that $i^g = j$. This implies that $\xi_{i_3}^{(\gamma \alpha)}(g) = \xi_{j_3}$, and so the induced action of $G_{\alpha 0\beta}$ on $H_\beta \setminus \{1\}$ is transitive. On the other hand, $H_\beta$ is a $p$-group, and so for some $z \in \Delta_2 \setminus \{0\}$, $\xi_{i_3} \in Z(H_\beta)$. The center is a characteristic subgroup, and therefore $H_\beta$ is elementary abelian.

**Proposition 2.8.** The stabilizer of a point in $G_{\alpha \beta}^{\Delta_2}$ is involved in Aut($H$).

**Proof.** Suppose $g \in G_{\alpha 0\beta}$ and $\varphi_{\alpha \beta}(g) = 1$. In particular, for all $i \in \Delta_2 \setminus \{0\}$, $\xi_{i_3} = \xi_{i_3}^{(\gamma \alpha)} = \xi_{i_3}^{(\gamma \alpha)}$. By Lemma 2.4, this implies that $i^g = i$ for all $i \in \Delta_2$, and so $g \in G_{\alpha 0\beta}^{[1]}$. Hence $\text{Ker}(\varphi_{\alpha \beta}) \leq G_{\alpha 0\beta}^{[1]}$, and so

$$G_{\alpha 0\beta}^{\Delta_2} \cong \varphi_{\alpha \beta}(G_{\alpha 0\beta}) / \varphi_{\alpha \beta}(G_{\alpha 0\beta}^{[1]}).$$
is involved in Aut(H).

**Corollary 2.9.** If the group $T$ has a nontrivial induced action on $H$, then $G^{\Delta_2}_{\alpha,\beta}$ is involved in Aut($H$).

**Proof.** Let $S = \text{soc}(G^{\Delta_2}_{\alpha,\beta})$ and $K = G^{\Delta_2}_{\alpha,\beta}$. Since $S$ is transitive on $\Delta_2$ and $K$ is the stabilizer of a point, $G^{\Delta_2}_{\alpha,\beta} = SK$, and hence, by Propositions 2.5, 2.6 and 2.8,

$$G^{\Delta_2}_{\alpha,\beta} = SK \cong \left(\varphi_{\alpha,\beta}(T)/\varphi_{\alpha,\beta}(G^{[1]}_{\alpha,\beta})\right) \left(\varphi_{\alpha,\beta}(G_{\alpha,\beta})/\varphi_{\alpha,\beta}(G^{[1]}_{\alpha,\beta})\right) \cong \varphi_{\alpha,\beta}(G_{\alpha,\beta})/\varphi_{\alpha,\beta}(G^{[1]}_{\alpha,\beta}).$$

We will next show that there exists a locally 3-arc-transitive regular cover of $K_{m,n}$ only if there exist locally 3-arc-transitive regular covers of $K_{2,m}$ and $K_{2,n}$, respectively. For a complete bipartite graph $K_{m,n}$ with $|\Delta_1| = m$ and $|\Delta_2| = n$, if $\gamma, \delta \in \Delta_1$, then we define $K^{\gamma,\delta}_{m,n} \cong K_{2,n}$ to be the subgraph induced on $\{\gamma, \delta\} \cup \Delta_2$. The following proof, which simultaneously simplifies and strengthens results from a previous version of this paper, comes from an anonymous referee:

**Proposition 2.10.** Let $P : \tilde{\Gamma} \to K_{m,n}$ be a regular covering projection, and assume that $\tilde{\Gamma}$ is locally $(G,3)$-arc-transitive, where $G$ is the lift of a group $G \leq \text{Aut}(K_{m,n})$. If $\tilde{\alpha} \in P^{-1}(\alpha)$ and $\tilde{\beta} \in P^{-1}(\beta)$ are vertices in the same connected component $\tilde{\Sigma}$ of $P^{-1}(K^{\alpha,\beta}_{m,n})$, then $\tilde{\Sigma}$ is locally $(\tilde{G}_{\{\tilde{\alpha},\tilde{\beta}\}},3)$-arc-transitive.

**Proof.** It is clear that the graphs $K^{\gamma,\delta}_{m,n} \cong K_{2,n}$ are isomorphic for each choice of $\gamma, \delta \in \Delta_1$. Moreover, given a vertex $0 \in \Delta_2$, the group $G_0$ is 2-transitive on $\Delta_1$, and so for any $\gamma, \delta \in \Delta_1$, there is a subgroup of $G$ that takes $K^{\alpha,\beta}_{m,n}$ to $K^{\gamma,\delta}_{m,n}$. Since this group lifts by assumption, there is an isomorphism $\kappa : P^{-1}(K^{\gamma,\delta}_{m,n}) \to P^{-1}(K^{\alpha,\beta}_{m,n})$ by Lemma 2.2. Furthermore, the subgroup of $G$ for which $K^{\alpha,\beta}_{m,n}$ is invariant, namely $G_{\{\alpha,\beta\}}$, is a locally 3-arc-transitive subgroup of $\text{Aut}(K^{\alpha,\beta}_{m,n})$. Since $G_{\{\alpha,\beta\}}$ lifts, by [5, Theorem 3], any connected component of $P^{-1}(K^{\alpha,\beta}_{m,n})$ is also locally $(G,3)$-arc-transitive. The result follows.

Proposition 2.10 implies that, for a complete bipartite graph $K_{m,n}$ with a locally 3-arc-transitive regular cover, if $|\Delta_1| = m$, $|\Delta_2| = n$, $\beta \in \Delta_1$, and $H_\beta$ is the group generated by the voltages assigned to the edges incident with $\beta$, then there exists a locally 3-arc-transitive regular cover of $K_{2,n}$ with covering transformation group isomorphic to $H_\beta$. Moreover, the voltage assigned to the arc $(i, \beta)$ in $K_{2,n}$ is the same as that which is assigned to the arc $(i, \beta)$ in $K^{\alpha,\beta}_{m,n}$, i.e., there is an isomorphism of covering projections. Note further that by Lemma 2.2 for any vertices $\beta, \gamma \in \Delta_1 \setminus \{\alpha\}$, there is a group isomorphism $\tau : H_\gamma \to H_\beta$, and the group $H_\beta$ is independent of the choice of $\beta$. We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 2.10 if $|\Delta_1| = n$, then $K_{2,n}$ has a locally $(\tilde{K},3)$-arc-transitive regular cover with covering transformation group $H_\beta$, where $\tilde{K}$ is a lift of the group $K \leq \text{Aut}(K_{2,n})$. Let $\Delta$ be the bipart of size $n$ in $K_{2,n}$. Note that $K_{\alpha,\beta} = K^{\Delta}_{\alpha,\beta}$, so $K^{[1]}_{\alpha,\beta} = 1$. Let $T = \text{soc}(K_{\alpha,\beta})$ and $\varphi_{\alpha,\beta} : K_{\alpha,\beta} \to \text{Aut}(H_\beta)$ be the induced action, as above. If $\varphi_{\alpha,\beta}(T) = 1$, then, by Proposition 2.7, $n = p^j$ for some prime $p$ and $f \in \mathbb{N}$, and $H_\beta$ is an elementary abelian group of order $p^f$. Otherwise, $T$ has a nontrivial induced action on $H_\beta$, and, by Corollary 2.9 $K_{\alpha,\beta} = K^{\Delta}_{\alpha,\beta}$ is involved in $\text{Aut}(H_\beta)$. Moreover, since $K^{[1]}_{\alpha,\beta} = 1$, we have $\varphi_{\alpha,\beta}(K^{[1]}_{\alpha,\beta}) = 1$, and

$$K_{\alpha,\beta} \cong \varphi_{\alpha,\beta}(K_{\alpha,\beta})/\varphi_{\alpha,\beta}(K^{[1]}_{\alpha,\beta}) \cong \varphi_{\alpha,\beta}(K_{\alpha,\beta}).$$
In this case, \( K_{\alpha \beta} \), which has a 2-transitive action on \( n = |\Delta| = |\Delta_i| \) elements, is isomorphic to a subgroup of \( \text{Aut}(H_\beta) \). Using Proposition 2.8, a similar argument shows that, in either case, \( K_{\alpha \beta}^{\Delta_i} \cong G^{\Delta_i}_{\alpha \beta v} \) is isomorphic to a subgroup of \( \text{Aut}(H_\beta) \) for any \( v \in \Delta_i \). The result follows. \( \square \)

3. Locally 3-arc-transitive regular covers of complete bipartite graphs with cyclic covering transformation group

Let \( \Gamma = K_{m,n} \) have biparts \( \Delta_1 \) and \( \Delta_2 \), and assume that \( \Gamma \) has a locally 3-arc-transitive regular cover \( \hat{\Gamma} \) with covering projection \( \mathcal{P} : \hat{\Gamma} \to \Gamma \). We will also assume that the group of covering transformations is cyclic, i.e., let \( H = \text{CT}(\mathcal{P}) = (\xi) \cong \mathbb{Z}_N \) for some integer \( N \in \mathbb{N} \), \( N > 1 \), where the cyclic group will be viewed additively. As in the previous section, we fix \( \beta \neq \alpha \in \Delta_1 \), let \( G \) be the group of automorphisms that lift, and let \( T = T_{\alpha \beta} \) be the subgroup of \( G_{\alpha \beta} \) that contains \( G^{[1]}_{\alpha \beta} \) and \( T/G^{[1]}_{\alpha \beta} \cong \text{soc}(G^{\Delta_i}_{\alpha \beta}) \). Furthermore, \( G_{\alpha \beta} \) has an induced action on \( H \) given by \( \varphi_{\alpha \beta} : G_{\alpha \beta} \to \text{Aut}(H) \).

**Lemma 3.1.** If \( |\Delta_2| > 2 \), then the induced action of the group \( T \) on \( H \) is trivial, i.e., \( \varphi_{\alpha \beta}(T) = 1 \).

**Proof.** By Theorem 1.1, if the induced action of \( T \) on \( H \) is not trivial, then there exists \( X \leq \text{Aut}(H_\beta) \) with a 2-transitive action on \( |\Delta_2| \) elements. On the other hand, \( H \) is cyclic and generated by the voltages assigned to the edges of \( \Gamma \), and so \( H_\beta \leq H \) is also cyclic. (Actually, by Lemma 2.4, the voltages assigned to the edges of \( \Gamma \) all have the same order, so \( H_\beta = H \).) Thus \( \text{Aut}(H_\beta) \) is isomorphic to the group of units of \( H_\beta \), which is abelian. When \( |\Delta_i| > 2 \), no 2-transitive group on \( \Delta_i \) is abelian, and so no such group \( X \) can exist, a contradiction. \( \square \)

**Lemma 3.2.** If \( |\Delta_2| > 2 \), then \( G^{\Delta_i} = \mathbb{Z}_p \mathbb{Z}_{p-1} \) and \( H \cong \mathbb{Z}_p \).

**Proof.** By Lemma 3.1 the group \( T \) has trivial induced action on \( H \), i.e., \( T \leq \text{Ker}(\varphi_{\alpha \beta}) \). By Theorem 1.1, the group of covering transformations \( H \) must be isomorphic to \( \mathbb{Z}_p^f \), where \( p \) is a prime and \( f \in \mathbb{N} \). Since \( H \) is cyclic, \( H \cong \mathbb{Z}_p \) and \( |\Delta_2| = p \). Moreover, by Proposition 2.7, \( \text{soc}(G^{\Delta_i}_{\alpha \beta}) \cong \mathbb{Z}_p \), and hence \( G^{\Delta_i}_{\alpha \beta} \cong \mathbb{Z}_p : \mathbb{Z}_{p-1} \). \( \square \)

**Proposition 3.3.** The only possibilities for \( \Gamma = K_{m,n} \) are \( K_{2,p} \) and \( K_{p,p} \), where \( p \) is a prime.

**Proof.** By Lemma 3.2, we may assume that \( |\Delta_2| = n = p \) for some prime \( p \). On the other hand, using the same reasoning, \( |\Delta_1| = p' \) for some prime \( p' \). If both \( p, p' > 2 \), then by Lemma 3.2 it is impossible for all edges to have voltages from both \( \mathbb{Z}_p \) and \( \mathbb{Z}_{p'} \) unless \( p = p' \). Hence the only possibilities are \( K_{2,p} \) and \( K_{p,p} \), where \( p \) is a prime. \( \square \)

**Proof of Theorem 1.2.** First, we note that Proposition 3.3 implies that \( K_{2,2}, K_{2,p}, \) and \( K_{p,p} \) are the only three possibilities; all that remains is to show that unique covers actually exist in each listed case.

We note that \( K_{2,2} \) is a four cycle, and the cover of \( K_{2,2} \) derived from the group \( \mathbb{Z}_N \) is a \( 4N \)-cycle. Moreover, for any two covering transformation groups \( H_1 \cong H_2 \cong \mathbb{Z}_N \), there is an obvious isomorphism, and so by Lemma 2.2 there is a unique such cover up to isomorphism of covering projections.

Suppose now that \( |\Delta_1| = m = 2 \) and \( |\Delta_2| = n = p \), where \( p \) is an odd prime. By Lemma 3.2 \( N = p \). Continuing with the notation above, let \( \Delta_1 = \{\alpha, \beta\} \) and \( \Delta_2 = \{0, 1, ..., p-1\} \). By Lemma 2.4, the voltages assigned to the co-tree arcs \((i, \beta)\), where \( 1 \leq i \leq p-1 \), must be distinct non-identity elements of \( \mathbb{Z}_p \). Up to relabeling the vertices, i.e., up to an automorphism of \( \Gamma = K_{2,p} \), there is a unique way to assign voltages.
to the edges. By Lemma 2.2 there is at most one such cover of \( K_{2,p} \) up to isomorphism of covering projections.

It only remains to show that this covering projection has the desired properties. Let \( H = \langle \xi \rangle \), viewed additively, and without a loss of generality assume that \( \xi_{i\beta} = i\xi \) for all \( i \). Note that the closed walks based at the vertex \( \alpha \) are generated by the closed walks of the form \( (\alpha, i, \beta, 0, \alpha) \), \( 1 \leq i \leq p-1 \). Thus by Lemma 2.3 an automorphism \( g \) of \( \Gamma = K_{2,p} \) lifts to \( \tilde{g} \) if and only if there exists a solution to the equations \( \xi_{i\beta}^g = \xi_{W_i}^g \) for all \( i \), where \( W_i = (\alpha, i, \beta, 0, \alpha) \) and \( g^# \in \text{Aut}(H) \).

Given a primitive element \( t \) of the field \( GF(p) \), let \( g \) be the \((p-1)\)-cycle \( (1 \ t \ 2 \ \ldots \ t^{p-2}) \) in \( \text{Sym}(\Delta_2) \), and assume that \( g \) fixes both \( \alpha \) and \( \beta \). Then, for all \( i > 0 \), assuming that \( i \equiv t^k \mod p \), \( \xi_{i\beta}^g = \xi_{W_i}^g = (t^k \cdot \xi)^g \); on the other hand, \( \xi_{W_i}^g = \xi_{(i\beta)g\beta} = (t^k)^g \xi = t^{k+1}\xi \). Indeed, these equations are satisfied by the outer automorphism \( \nu \) of \( H \) given by \( \nu : \xi \mapsto t\xi \), and so the \((p-1)\)-cycle \( g \) lifts.

Let \( h \) be the \( p \)-cycle \( (0 \ 1 \ \ldots \ (p-1)) \) in \( \text{Sym}(\Delta_2) \). Then \( \xi_{i\beta}^h = (i\xi)^h \); on the other hand, \( \xi_{W_i}^h = -\xi_{i\beta} + \xi_{(i+1)\beta} = -\xi + (i+1)\xi = i\xi \). These equations are satisfied by the trivial automorphism of \( H \), and so \( h \) lifts. The group generated by \( h \) and \( g \) acts 2-transitively on \( \Delta_2 \) while leaving all of \( \Delta_1 \) fixed.

Next, let \( x \) be the automorphism of \( K_{2,p} \) that interchanges \( \alpha \) and \( \beta \) but fixes all other vertices. We will now associate the voltage \( \xi_{i\beta} \) with the closed walk \( U_i := (0, \alpha, i, \beta, 0) \) based at 0. In order for \( x \) to lift, there must exist \( x^# \in \text{Aut}(H) \) such that \( \xi_{i\beta}^# = \xi_{U_i}^# = \xi_{i\beta}^{-1} \) for all \( i \in \Delta_2 \setminus \{0\} \). This corresponds to the outer automorphism \( \xi : \xi \mapsto -\xi \), and so \( x \) lifts as well. The group generated by \( x \) acts 2-transitively on \( \Delta_1 \) while leaving all of \( \Delta_2 \) fixed. It is easy to see that the group \( G = (g, h, x) \) lifts and \( \Gamma \) is locally \((G, 3)\)-arc-transitive. By [5, Theorem 3], this unique cover (up to isomorphism of covers) will be a locally 3-arc-transitive graph.

Finally, we suppose that \( m = n = p \), where \( p \) is an odd prime. By Lemma 3.2 \( H \cong \mathbb{Z}_p \). By Lemma 2.4 the voltages assigned to the edges incident with a single vertex are distinct and all come from \( \mathbb{Z}_p \). Up to relabeling the vertices, i.e., up to an automorphism of \( K_{p,p} \), there is a unique way to do this. Namely, if we let \( \Delta_1 = \{\alpha_0, \alpha_1, ..., \alpha_{p-1}\} \), let \( \Delta_2 = \{0, 1, ..., p-1\} \), and assume that the spanning tree with identity voltages consists of all edges incident with either \( \alpha_0 \) or 0, we will assign voltages to co-tree edges by letting \( \xi_{i\alpha_j} \) be the \((i, j)\)-entry of the following \((p-1) \times (p-1)\) matrix:

\[
\begin{pmatrix}
\xi & 2\xi & 3\xi & \ldots & (p-1)\xi \\
2\xi & 2(2\xi) & 3(2\xi) & \ldots & (p-1)(2\xi) \\
3\xi & 2(3\xi) & 3(3\xi) & \ldots & (p-1)(3\xi) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(p-1)\xi & 2((p-1)\xi) & 3((p-1)\xi) & \ldots & (p-1)((p-1)\xi)
\end{pmatrix}
\]

It is clear that each row and each column of the matrix contains each nonidentity element of \( \langle \xi \rangle \) exactly once. Hence we have a voltage assignment which satisfies all the above lemmas, and, up to permuting the vertices of \( K_{p,p} \), this assignment is unique. Arguing again as in the \( K_{2,p} \) case, we see that the desired automorphisms lift, and so we again get a cover of the desired type that is unique up to isomorphism of covering projections. \( \square \)
4. Locally 3-arc-transitive regular covers of complete bipartite graphs with elementary abelian covering transformation group

In this section, we will assume that the group of covering transformations is elementary abelian; that is, we will assume that \( H := \text{CT}(P) \cong \mathbb{Z}_p^d \), where \( p \) is a prime and \( d \geq 2 \) (the case \( d = 1 \) falls under the cyclic case). We begin by proving Theorem 4.1 which is essentially a corollary of Theorem 1.1.

**Proof of Theorem 4.1.** In this case, \( H_\beta \cong \mathbb{Z}_p^f \) for some \( f \leq d \). Thus \( \text{Aut}(H_\beta) \cong \text{GL}_f(p) \leq \text{GL}_d(p) \), and the result follows. \( \square \)

By Theorem 1.1, the process of finding the locally 3-arc-transitive covers of \( K_{m,n} \) comes down to finding locally 3-arc-transitive covers of \( K_{2,m} \) and \( K_{2,n} \). While it can be extremely difficult to determine which \( K_{2,m} \) have lifts in full generality, there are certain cases that are not so difficult.

**Proposition 4.1.** Up to isomorphism of covering projections, \( \Gamma = K_{p^f} \) has a unique locally 3-arc-transitive regular cover \( \tilde{\Gamma} \) with covering projection \( P : \tilde{\Gamma} \to \Gamma \) such that \( H = \text{CT}(P) \cong \mathbb{Z}_p^f \).

**Proof.** By Lemma 2.3, a different nonidentity element of \( H \) must be assigned as a voltage to each co-tree edge of \( \Gamma \). Up to relabeling the vertices, i.e., up to an automorphism of \( \Gamma \), there is a unique way to do this, and so, by Lemma 2.2, there is at most one such cover.

Let the biparts of \( \Gamma \) be \( \Delta_1 = \{\alpha, \beta\} \) and \( \Delta_2 = \{0, \ldots, p^f-1\} \). Define \( W_i \) to be the closed walk \((\alpha, i, \beta, 0, \alpha), 1 \leq i \leq p^f - 1\). The \( W_i \) form a generating set for all closed walks based at \( \alpha \). We will show that this particular voltage assignment results in a locally 3-arc-transitive regular cover \( \tilde{\Gamma} \). First, let \( h \in \text{Aut}(\Gamma) \) be the automorphism that fixes \( \Delta_2 \) but interchanges \( \alpha \) and \( \beta \). Thus

\[ \xi_{W_i}^h \leftrightarrow (\alpha, i, \beta, 0, \alpha)^h = (\beta, i, \alpha, 0, \beta) \leftrightarrow -\xi_{i\beta} = -\xi_{W_i}. \]

Since there is always an automorphism inverting all elements of an abelian group, by Lemma 2.3, \( h \) lifts.

We will view \( H \) additively as a vector space, and we label the elements of \( H \) as \( \overrightarrow{0}, \ldots, (p^f - 1) \), where \( \overrightarrow{i} \) is the voltage of the arc \((i, \beta)\). The \( W_i \) generate a set of closed walks based at \( \alpha \), and the voltage of \( W_i \) is \( \overrightarrow{i} \). Suppose \( g \in \text{Aut}(\Gamma) \) such that \( g \) fixes \( \alpha \), \( \beta \), and \( 0 \). Then

\[ \xi_{W_i}^g \leftrightarrow (\alpha, i, \beta, 0, \alpha)^g = (\alpha, i^g, \beta, 0, \alpha) \leftrightarrow \overrightarrow{i^g}. \]

Thus any \( g \in \text{Aut}(\Gamma) \) that acts on \( \{1, \ldots, p^f - 1\} \) as an element of \( \text{GL}_f(p) \) acts on \( \{\overrightarrow{1}, \ldots, \overrightarrow{p^f - 1}\} \) will lift to an automorphism of \( \tilde{\Gamma} \).

Now, let \( g_i, 0 \leq i \leq p^f - 1 \), be defined to be the permutation of \( \{0, \ldots, p^f - 1\} \) such that \( 0^{g_i} = i \) and \( j^{g_i} = k \), where \( \xi_{k\beta} \) has voltage corresponding to \( \overrightarrow{i} + \overrightarrow{j} \). Since the voltages on edges incident with \( \beta \) are all distinct, this is well-defined permutation. Hence

\[ \xi_{W_i}^{g_i} \leftrightarrow (\alpha, j, \beta, 0, \alpha)^{g_i} = (\alpha, k, \beta, i, \alpha) \leftrightarrow -\overrightarrow{i} + \overrightarrow{i} + \overrightarrow{j} = \overrightarrow{j} = \xi_{W_j}. \]

Since \( \xi_{W_j}^{g_i} = \xi_{W_j} \) is satisfied for all \( j \) by letting \( g_i^# \) be the trivial automorphism of \( H, g_i \) lifts. Therefore, a group of automorphisms \( X \) of \( \Gamma \) stabilizing each element in \( \Delta_1 \) that acts isomorphically to \( \text{AGL}_f(p) \) on \( \Delta_2 \) lifts. Since the group \( G = \langle X, h \rangle \) lifts, and \( \Gamma \) is locally \((G, 3)\)-arc-transitive, \( \tilde{\Gamma} \) is locally \((\tilde{G}, 3)\)-arc-transitive, as desired. \( \square \)
Corollary 4.2. Let $H \cong \mathbb{Z}_p^d$. The graph $K_{2,p}$ has a locally 3-arc-transitive regular cover with $CT(\mathcal{P}) \leq H$ if and only if $1 \leq f \leq d$.

Proof. If $f > d$, then there are not enough distinct elements of $H$ to assign as voltages to arcs of $K_{2,p}$, a contradiction. If $f \leq d$, then we choose a subgroup $K \leq H$ such that $K \cong \mathbb{Z}_p^f$. The result follows from Proposition 4.1.

Combined with Theorem 1.3, Proposition 4.1 shows that finding the possible $m$ for which $K_{2,m}$ has a locally 3-arc-transitive regular cover with elementary abelian covering transformations isomorphic to $\mathbb{Z}_p^f$ comes down to representation theory over $\text{GF}(p)$. We will now use these results to classify the complete bipartite graphs with locally 3-arc-transitive regular covers whose covering transformation group is a subgroup of $\mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is a prime.

**Lemma 4.3.** Let $p, r$ be distinct primes. If $\mathbb{Z}_p^f \leq \text{GL}_d(p)$, then $f \leq d$.

Proof. This is a basic exercise in representation theory and is left to the reader.

**Proposition 4.4.** Let $H \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is a prime. The only values of $m$ for which the complete bipartite graph $K_{2,m}$ has a locally 3-arc-transitive regular cover whose covering transformations are a subgroup of $H$ are $m = 2, 3, p$, and $p^2$.

Proof. Let the biparts of $\Gamma = K_{2,m}$ be $\Delta_1 = \{\alpha, \beta\}$ and $\Delta_2$. Let $G \leq \text{Aut}(\Gamma)$ be the subgroup of automorphisms that lifts to $\tilde{\Gamma}$ such that $\Gamma$ is locally $(\tilde{\Gamma}, 3)$-arc-transitive. By Theorem 4.3, if $S := \text{soc}(G_{\alpha\beta}^{\Delta_2})$, then either $S \cong \mathbb{Z}_p^f$ for some $f \in \mathbb{N}$ or $G_{\alpha\beta}^{\Delta_2} \cong \text{GL}_2(p)$. By Corollary 4.2, if $S \cong \mathbb{Z}_p^f$ then $f = 1, 2$, and both $K_{2,p}$ have such covers, which are unique up to isomorphism of covering projections by Proposition 4.1. We also know that $K_{2,2}$ has such a cover by Theorem 1.2. By [21, Proposition 6.4], $K_{2,3}$ has a unique such cover (up to isomorphism of covering projections), and all of $\text{Aut}(K_{2,3})$ lifts.

Assume now that $G_{\alpha\beta}^{\Delta_2} \cong \text{GL}_2(p)$ and $|\Delta_2| > 3$. By [13, Proposition 5.5.10], no nonabelian finite simple group has a nontrivial representation of degree 2 in odd characteristic, and since $|\text{GL}_2(2)| = 6$, no nonabelian finite simple group is isomorphic to a subgroup of $\text{GL}_2(2)$. Hence $S \cong \mathbb{Z}_p^f$ for some prime $r \neq p$, and, by Lemma 4.3, $f \leq 2$.

Suppose first that $f = 2$. Let $G_{\alpha \beta} \cong X \leq \text{GL}_2(p)$, and let $X = NK$, where $N \cong \text{soc}(G_{\alpha \beta}^{\Delta_2})$ and $K \cong G_{\alpha \beta \epsilon}$, where $\epsilon \in \Delta_2$. Since $N$ is a minimal normal subgroup of $X$ and $\text{GF}(p)^*$ is cyclic, $N$ is in the kernel of the determinant map. Moreover, the center of $\text{SL}_2(p)$ has order 2, so either $N \cong \mathbb{Z}_2^2$ or $N \leq \text{PSL}_2(p)$. The only subgroups of $\text{PSL}_2(p)$ isomorphic to $\mathbb{Z}_2^2$, $r \neq p$, are isomorphic to $\mathbb{Z}_2^2$. Hence $N \cong \mathbb{Z}_2^2$ and so $G_{\alpha \beta} \cong A_4$. However, $A_4$ has no faithful representation of degree 2, a contradiction. Hence $f = 1$, and $G_{\alpha \beta} \cong \mathbb{Z}_r \times \mathbb{Z}_{r-1} \cong \text{AGL}_1(r)$, where $r > 3$. However, when $r-1 > 2$, the group $\mathbb{Z}_r : \mathbb{Z}_r^{-1}$ has no faithful representation of degree 2 (see, for instance, [17, Theorem 25.10]), a final contradiction.

Once the locally 3-arc-transitive regular covers of $K_{2,m}$ and $K_{2,n}$ have been found, it remains to be seen whether $K_{m,n}$ has a locally 3-arc-transitive regular cover. In certain cases, we can give a definitive answer.

**Proposition 4.5.** If $\Gamma = K_{p^f, p^d}$, where $f$ divides $d$, then there exists a regular cover $\tilde{\Gamma}$ and a covering projection $\mathcal{P} : \tilde{\Gamma} \to \Gamma$ such that $\tilde{\Gamma}$ is locally 3-arc-transitive and $CT(\mathcal{P}) \cong \mathbb{Z}_{p^f}^d$.

Proof. Let $V = \mathbb{Z}_{p^f}^d$ and let $\tau$ be a generator of $\text{GL}_1(p^d) \leq \text{GL}_d(p)$. Since $f$ divides $d$, $p^f - 1$ divides $p^d - 1$, and hence $\sigma = \tau_{p^f - 1}^{-1}$ has order $p^f - 1$, stabilizes a decomposition...
of \( V \) into \((p^d-1)/(p^f-1)\) subspaces of size \( p^f-1 \), and acts semiregularly on the non-zero elements of \( V \).

Let the biparts of \( \Gamma \) be \( \Delta_1 = \{\gamma_0, \ldots, \gamma_{p^f-1}\} \) and \( \Delta_2 = \{\delta_0, \ldots, \delta_{p^d-1}\} \). We choose a nonzero element \( \xi \in V \), and to each arc \((\delta_i, \gamma_j)\), where \( 1 \leq i \leq p^d-1 \) and \( 1 \leq j \leq p^f-1 \), we assign the voltage \( \xi_{ij} = \xi_{ij}^{\sigma^j} = \xi^{\sigma_j} \). Note that for any fixed \( i \), the voltages \( \xi_{ij} \) are distinct for \( j \neq j' \), and, similarly, for any fixed \( j \), the voltages \( \xi_{ij} \) and \( \xi_{i'j} \) are distinct for \( i \neq i' \). We define further the closed walk \( W_{ij} = (\gamma_0, \delta_i, \gamma_j, \delta_0, \gamma_0) \) based at \( \gamma_0 \). Let \( \delta_0, \delta_j, \gamma_j \) and \( \delta_k \) be fixed elements of \( \Delta_1 \), and \( \delta_{0k}, \alpha, \beta, \gamma \) a walk \((\delta_0, \gamma, \delta_j, \delta_0, \gamma_0) \) based at \( \delta_0 \).

We will show next that the element \( g \in \text{Aut}(\Gamma) \) lifts, where \( g \) acts like the cycle \((\delta_1 \delta_2 \ldots \delta_{p^d-1})\) on \( \Delta_2 \) and trivially on \( \Delta_1 \). In this case, we need to find \( g^\# \in \text{Aut}(V) \) that satisfies:

\[
\left(\xi^{\tau \sigma_j}\right)^g = \xi_{ij}^g = \xi_{1ij} \leftrightarrow (\gamma_0, \delta_i, \gamma_j, \delta_0, \gamma_0)^g
\]

and \( g^\# = \tau \) demonstrates that \( g \) lifts.

We will show next that the element \( h \in \text{Aut}(\Gamma) \) lifts, where \( h \) acts like the cycle \((\gamma_1 \gamma_2 \ldots \gamma_{p^f-1})\) on \( \Delta_1 \) and trivially on \( \Delta_2 \). In this case, we need to find \( h^\# \in \text{Aut}(V) \) that satisfies:

\[
\left(\xi^{\tau \sigma_j}\right)^h = \xi_{ij}^h = \xi_{1ij} \leftrightarrow (\gamma_0, \delta_i, \gamma_j, \delta_0, \gamma_0)^h
\]

and \( h^\# = \sigma \) demonstrates that \( h \) lifts.

We will next show that a group \( X \) acting regularly on \( \Delta_2 \) and trivially on \( \Delta_1 \) lifts. We will define the element \( x_k \in X \) that sends \( \delta_{0k} \) to \( \delta_k \). We then define \( \delta_{ik} = \delta_{ik}^{\tau \sigma_j} = \delta_{ik} \), where \( \xi_{ij} = \xi_{ij} + \xi_{kj} \). In this case, we need \( x_k^\# \in \text{Aut}(V) \) that satisfies:

\[
\xi_{ij}^{x_k^\#} = \xi_{ij}^{x_k} \leftrightarrow (\gamma_0, \delta_i, \gamma_j, \delta_0, \gamma_0)^{x_k} = (\gamma_0, \delta_i, \gamma_j, \delta_k, \gamma_0) \leftrightarrow -\xi_{kj} + \xi_{ij} = \xi_{ij},
\]

and \( x_k^\# = 1 \in \text{Aut}(V) \) satisfies these equations, and so \( x_k \) lifts. A group \( Y \) acting regularly on \( \Delta_1 \) and trivially on \( \Delta_2 \) is shown to lift analagously (the walks \( U_{ij} \) are a generating set for the closed walks based at \( \delta_0 \)). The group that lifts acts locally 3-arc-transitively on \( \Gamma \), and hence \( \tilde{\Gamma} \) is locally 3-arc-transitive as well.

\textbf{Lemma 4.6.} Let \( p \) be a prime. If \( \Gamma = K_{d,p^2} \), then there exists a regular cover \( \tilde{\Gamma} \) of \( \Gamma \) with covering projection \( \mathcal{P} : \tilde{\Gamma} \to \Gamma \) such that \( \tilde{\Gamma} \) is locally 3-arc-transitive and \( \text{CT}(\mathcal{P}) \cong \mathbb{Z}_p \times \mathbb{Z}_p \).

\textbf{Proof.} First, if \( p = 3 \), then the result follows from Proposition 4.5 Let \( \tau \) be a generator of \( \text{GL}_1(p^2) \). If \( p \neq 3 \), then \( p^2 = 1 \mod 3 \), and so we can find \( \sigma = \tau^{p^2-1} \) of order 3. Note also that since \( \sigma \in \langle \tau \rangle, \sigma, \tau \rangle = 1 \). We define the biparts of \( \Gamma \) to be \( \Delta_1 = \{\alpha, \beta, \gamma\} \) and \( \Delta_2 = \{0, 1, \ldots, p^2-1\} \). If \( V = \mathbb{Z}_p \times \mathbb{Z}_p \) and \( \xi \) is a nonidentity element of \( V \) such that \( \xi^\sigma \not\in \langle \xi \rangle \), we define the voltage on the arc \((i, \beta)\) to be \( \xi_{\alpha j} = \xi_{ij}^{\tau} \) and the voltage on the arc \((i, \gamma)\) to be \( \xi_{i \gamma} = \xi_{ij}^{\sigma} + \xi_{ij}^{\tau} \). We define \( W_{ik} \) to be the closed walk \((\alpha, i, \epsilon, 0, \alpha) \) and \( U_{ik} \) to be the closed walk \((0, \alpha, i, \epsilon, 0) \).
We will show first that the element \( g \in \text{Aut}(\Gamma) \) lifts, where \( g \) acts like the cycle \((1 \ 2 \ \ldots \ \ p^n - 1)\) on \( \Delta_2 \) and trivially on \( \Delta_1 \). In this case, we need to find \( g^\# \in \text{Aut}(V) \) that satisfies:

\[
\left( \xi^{x^i} \right)^{g^\#} = \xi_{W_{i\beta}}^{n} = \xi_{W_{i\beta}}^g \leftrightarrow (\alpha, i, \beta, 0, \alpha)^g = (\alpha, (i + 1) \mod (p^n - 1), \beta, 0, \alpha) \leftrightarrow \xi^{x^i+1} = \left( \xi^{x^i} \right)^{\tau},
\]

\[
\left( \xi^{x^i + \xi^{x^i \sigma}} \right)^{g^\#} = \xi_{W_{i\gamma}}^{n} = \xi_{W_{i\gamma}}^g \leftrightarrow (\alpha, i, \gamma, 0, \alpha)^g = (\alpha, (i + 1) \mod (p^n - 1), \gamma, 0, \alpha) \leftrightarrow \xi^{x^i+1} + \xi^{x^i \sigma} = \left( \xi^{x^i + \xi^{x^i \sigma}} \right)^{\tau},
\]

and hence \( g^\# = \tau \) demonstrates that \( g \) lifts.

We will show now that the element \( x \in \text{Aut}(\Gamma) \) lifts, where \( x \) acts like the cycle \((\alpha \ \beta \ \gamma)\) on \( \Delta_1 \) and trivially on \( \Delta_2 \). In this case, we need to find \( x^\# \in \text{Aut}(V) \) that satisfies:

\[
\left( \xi^{x^i} \right)^{x^\#} = \xi_{i\beta}^{x^\#} = \xi_{U_{i\beta}}^x \leftrightarrow (0, \alpha, i, \beta, 0)^x = (0, \beta, i, \gamma, 0) \leftrightarrow \xi_{i\gamma} - \xi_{i\beta} = \left( \xi^{x^i} \right)^{\sigma},
\]

\[
\left( \xi^{x^i + \xi^{x^i \sigma}} \right)^{x^\#} = \xi_{i\gamma}^{x^\#} = \xi_{U_{i\gamma}}^x \leftrightarrow (0, \alpha, i, \gamma, 0)^x = (0, \beta, i, \alpha, 0) \leftrightarrow -\xi_{i\beta}.
\]

Since for any \( \eta \in V \), \((\eta + \eta^σ + \eta^{σ^2})^σ = (\eta + \eta^σ + \eta^{σ^2})\), and \( σ \) fixes only the identity \( 0 \in V \), for any \( \eta \in V \) we have \(-\eta = \eta^σ + \eta^{σ^2}\). Thus,

\[
\left( \xi^{x^i + \xi^{x^i \sigma}} \right)^{x^\#} = -\xi_{i\beta} = \left( \xi^{x^i + \xi^{x^i \sigma}} \right)^{\sigma},
\]

and hence \( x^\# = σ \) demonstrates that \( x \) lifts.

We will show now that a group \( Y \) acting regularly on \( \Delta_2 \) and trivially on \( \Delta_1 \) lifts. We will define the element \( y_k \in Y \) that sends \( 0^k \) to \( k \). We then define \( i^{y_k} = ℓ \), where \( \xi_{i\beta} = \xi_{i\beta} + \xi_{kβ} \). In this case, we need \( y_k^\# \in \text{Aut}(V) \) that satisfies:

\[
\xi_{i\beta}^h \leftrightarrow (\alpha, i, \beta, 0, \alpha)^y_k = (\alpha, ℓ, \beta, k, α) \leftrightarrow -\xi_{kβ} + \xi_{i\beta} = \xi_{iβ},
\]

\[
\xi_{i\gamma}^h \leftrightarrow (\alpha, i, \gamma, 0, \alpha)^y_k = (\alpha, ℓ, γ, k, α) \leftrightarrow -\xi_{kγ} + \xi_{iγ} = (\xi_{iβ} - \xi_{kβ}) + (\xi_{ℓβ} - \xi_{kβ})^σ = \xi_{iγ},
\]

and hence \( y_k^\# = 1 \in \text{Aut}(V) \) demonstrates that \( y_k \) lifts.

Finally, we will show that an element \( h \) lifts, where \( h \) stabilizes \( 0, p^2 - 1, \alpha \) and satisfies \( β^h = γ, γ^h = β \). We will define the action of \( h \) on \( \Delta_2 \setminus \{0, p^2 - 1\} \) as follows. The elements \( ξ, ξ^σ \) form a basis for the vector space \( V \), and there is a linear transformation \( ρ \in \text{Aut}(V) \) such that \( ξ^ρ = ξ + ξ^σ \) and \( (ξ^σ)^ρ = -ξ^σ \). For any \( i \in \Delta_2 \setminus \{0, p^2 - 1\} \), we define \( i^h = ℓ \), where \( ξ_{iγ}^ρ = \left( ξ^{x^i + ξ^{x^i σ}} \right)^ρ = ξ^{x^i} \). This is a well-defined automorphism of \( Γ \), and to see that it lifts, we need to find \( h^\# \in \text{Aut}(V) \) such that:

\[
\left( \xi^{x^i} \right)^{h^\#} = \xi_{iβ}^{h^\#} = \xi_{W_{iβ}}^h \leftrightarrow (α, i, β, 0, α)^h = (α, ℓ, γ, 0, α) \leftrightarrow \xi_{ℓγ} = ξ^{x^i} + ξ^{x^i σ}
\]

\[
\left( \xi^{x^i + ξ^{x^i σ}} \right)^{h^\#} = \xi_{iγ}^{h^\#} = \xi_{W_{iγ}}^h \leftrightarrow (α, i, γ, 0, α)^h = (α, ℓ, β, 0, α) \leftrightarrow \xi_{ℓβ} = \left( ξ^{x^i} + ξ^{x^i σ} \right)^ρ.
\]
Now, since the elements $\xi, \xi^\sigma$ form a basis, there exist $a_i, b_i \in \mathbb{Z}_p$ such that $\xi^{r^i} = a_i\xi + b_i\xi^\sigma$. Hence,

$$\xi^{r^i} + \xi^{r^{i\sigma}} = a_i\xi + b_i\xi^\sigma + a_i\xi^\sigma + b_i\xi^{2\sigma} = a_i\xi + b_i\xi^\sigma + a_i\xi^\sigma + b_i(\xi - \xi^\sigma) = (a_i - b_i)\xi + a_i\xi^\sigma.$$  

This means that

$$\xi^{r^i} = (\xi^{r^i} + \xi^{r^{i\sigma}})^\rho = ((a_i - b_i)\xi + a_i\xi^\sigma)^\rho = (a_i - b_i)\xi - b_i\xi^\sigma,$$

and a similar calculation shows that

$$\left(\xi^{r^i}\right)^\rho = a_i\xi + (a_i - b_i)\xi^\sigma = \xi^{r^i} + \xi^{r^{i\sigma}}.$$  

Therefore, $h^{a_i} = \rho$ demonstrates that $h$ lifts. The group generated by these automorphisms both lifts and acts locally 3-arc-transitively on $\Gamma$, and therefore $\tilde{\Gamma}$ is a locally 3-arc-transitive regular cover of $\Gamma$. \hfill $\square$

**Lemma 4.7.** Let $p$ be a prime. If $\Gamma = K_{3,p}$, then there exists a regular cover $\tilde{\Gamma}$ of $\Gamma$ with covering projection $P : \tilde{\Gamma} \to \Gamma$ such that $\tilde{\Gamma}$ is locally 3-arc-transitive and $\text{CT}(\mathcal{P}) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. \hfill $\square$

**Proof.** First, if $p = 2$ or $p = 3$, then the result follows from [21, Proposition 6.4] or Theorem 1.2 respectively. Let $V = \mathbb{Z}_p \times \mathbb{Z}_p$, and let $\xi, \eta$ be two elements of $V$ such that $(\xi) \cap (\eta) = 0$, the identity element of $V$, and let $\Gamma = K_{3,p}$ have biparts $\Delta_1 = \{\alpha, \beta, \gamma\}$ and $\Delta_2 = \{0, 1, \ldots, p - 1\}$. For each $i \in \Delta_2 \setminus \{0\}$ (which is naturally identified with $\mathbb{Z}_p \setminus \{0\}$), we assign to the arc $(i, \beta)$ the voltage $\xi_{i\beta} = i\xi$, and we assign to the arc $(i, \gamma)$ the voltage $\xi_{i\gamma} = i\eta$. A group stabilizing $\Delta_1$ and acting like $\text{AGL}_1(p)$ on $\Delta_2$ lifts by a proof analogous to that of Theorem 1.2. The automorphism $\sigma \in \text{Aut}(V)$ defined by $\xi^\sigma = \eta - \xi$ and $\eta^\sigma = -\xi$ demonstrates that the automorphism $x \in \text{Aut}(\Gamma)$ that acts trivially on $\Delta_2$ and as the 3-cycle $(\alpha \beta \gamma)$ on $\Delta_1$ lifts, and the automorphism $\rho \in \text{Aut}(V)$ defined by $\xi^\rho = \eta$ and $\eta^\rho = \xi$ demonstrates that the automorphism $h \in \text{Aut}(\Gamma)$ that acts trivially on $\Delta_2$ and as the transposition $(\beta \gamma)$ on $\Delta_1$ lifts. Therefore, the cover $\tilde{\Gamma}$ derived from this voltage assignment is locally 3-arc-transitive, as desired. \hfill $\square$

In general, it is not guaranteed that $K_{m,n}$ has a locally 3-arc-transitive regular cover even if $K_{2,m}$ and $K_{2,n}$ both have locally 3-arc-transitive regular covers with covering transformation groups isomorphic to a subgroup of $\mathbb{Z}_p^d$, as the following lemma shows:

**Lemma 4.8 ([20, Proposition 3.3]).** Let $p \neq 3$ be a prime. Then $K_{3,3}$ does not have a locally 3-arc-transitive regular cover with covering transformation group $\mathbb{Z}_p \times \mathbb{Z}_p$.

We can now prove Theorem 1.4.\hfill $\square$

**Proof of Theorem 1.4.** First, the result follows for $K_{2,2}, K_{2,3}, K_{2,p}, K_{2,p^2}, K_{3,p}, K_{3,p^2}, K_{p,p}, K_{p,p^2}$, and $K_{p^2,p^2}$ by Theorem 1.2, Propositions 4.4 and 4.5, and Lemmas 4.6 and 4.7. The only other possibility is $K_{3,3}$, which is ruled out by Lemma 4.8.  \hfill $\square$

**Proof of Corollary 1.5.** By Theorem 1.4, we only must consider $K_{2,2}, K_{2,3}, K_{2,p}, K_{2,p^2}$, $K_{3,p}, K_{3,p^2}, K_{p,p}, K_{p,p^2}$, and $K_{p^2,p^2}$. However, $K_{2,2}$ is just $C_4$, which has only one co-tree edge in a spanning tree, and can be eliminated immediately. For $K_{2,3}$, the uniqueness follows from [21, Proposition 6.4], and for the other graphs the uniqueness follows from Proposition 2.7 and reasoning as in the proof of Theorem 1.2.  \hfill $\square$
It should be noted that all of the covers obtained in Theorems 1.2 and 1.4 have induced local 2-transitive actions that are affine groups. On the other hand, if $E$ is the edge set of a graph $\Gamma$ and $T$ is the edge set of a spanning tree of $\Gamma$, [21, Proposition 6.4] shows that assigning the elements of a basis of $\mathbb{Z}_p^{E\setminus T}$ to the co-tree edges of $\Gamma$ induces a regular cover such that the full automorphism group $\text{Aut}(\Gamma)$ lifts. On the other hand, a group $G$ that has a 2-transitive action on $m$ elements and a degree $d$ representation over $GF(p)$ does not actually guarantee that a voltage assignment from $\mathbb{Z}_p^d$ to the arcs of $K_{2,m}$ exists such that $G$ lifts.

Example 4.9. Even though $A_5 \lesssim \text{GL}(3,11)$, there is no voltage assignment from $\mathbb{Z}_{11}^3$ to $K_{2,5}$ such that $A_5$ lifts.

To see this, we can use the algorithm described in [21]. Indeed, the dual representation of $A_5$ on $H_1(K_{2,5};\mathbb{Z}_{11}) \cong \mathbb{Z}_{11}^4$ leaves no 3-dimensional space invariant, and hence no such cover can exist.

The cases when $d > 2$ can be approached similarly. Theorem 1.3 reduces the problem of finding the possible values of $m$ for which $K_{2,m}$ can have a locally 3-arc-transitive cover whose covering transformation group is isomorphic to a subgroup of $\mathbb{Z}_p^d$ to which 2-transitive groups can have a $d$-dimensional representation over $GF(p)$. From there, if $K_{2,m}$ and $K_{2,n}$ both have locally 3-arc-transitive covers with covering transformation group isomorphic to a subgroup of $\mathbb{Z}_p^d$, then the algorithm presented in [21] can be used to determine whether or not $K_{m,n}$ has a locally 3-arc-transitive cover with covering transformation group isomorphic to a subgroup of $\mathbb{Z}_p^d$.

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