Final review problems  
Math 213, Fall 2016

Name:

P1. Integrate the 3-form \( xy \, dx \wedge dy \wedge dz \) over the portion of the unit ball in the first octant.

Solution 1. Answer: \( \frac{1}{15} \)

The unit ball is the solid unit sphere, given by \( x^2 + y^2 + z^2 \leq 1 \) in Cartesian coordinates. We can parametrize the unit ball using standard spherical coordinates,

\[
F(\rho, \varphi, \theta) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi),
\]

where \( 0 \leq \rho \leq 1 \). Since we only want the portion in the first octant, we have \( 0 \leq \varphi \leq \frac{\pi}{2} \) and \( 0 \leq \theta \leq \frac{\pi}{2} \).

To evaluate the integral, we first compute the pullback

\[
F^*(xy \, dx \wedge dy \wedge dz).
\]

Recall that the pullback of \( dx \wedge dy \wedge dz \) by \( F \) is \( \rho^2 \sin \varphi \, d\rho \wedge d\varphi \wedge d\theta \). Hence,

\[
F^*(xy \, dx \wedge dy \wedge dz) = (\rho \sin \varphi \cos \theta)(\rho \sin \varphi \sin \theta)\rho^2 \sin \varphi \, d\rho \wedge d\varphi \wedge d\theta = \rho^4 \sin^3 \varphi \sin \theta \cos \theta \, d\rho \wedge d\varphi \wedge d\theta.
\]

So, we integrate

\[
\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^4 \sin^3 \varphi \sin \theta \cos \theta \, d\rho \, d\varphi \, d\theta = \frac{1}{5} \int_0^{\pi/2} \int_0^{\pi/2} \sin^3 \varphi \sin \theta \cos \theta \, d\varphi \, d\theta
\]

\[
= \frac{1}{5} \int_0^{\pi/2} \int_0^{\pi/2} (\sin \varphi - \sin \varphi \cos^2 \varphi) \sin \theta \cos \theta \, d\varphi \, d\theta
\]

\[
= \frac{1}{5} \left[ -\cos \varphi + \frac{1}{3} \cos^3 \varphi \right]_0^{\pi/2} \sin \theta \cos \theta \, d\theta
\]

\[
= \frac{2}{15} \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta
\]

\[
= \frac{2}{15} \left[ \frac{1}{2} \sin^2 \theta \right]_0^{\pi/2}
\]

\[
= \frac{1}{15}.
\]
P2. Let $X = (x^3, y^3, z^3)$ be a vector field on $\mathbb{R}^3$ and let the 2-surface $M$ be the portion of the cylinder $x^2 + y^2 = 1$ between $z = 0$ and $z = 2$. Verify the Divergence Theorem for $X$.

Solution 2. Answer: $3\pi$

One way to verify the divergence theorem is to show that we can compute the flux of $X$ across $M$ by directly evaluating the formula and by applying the theorem. To compute the flux directly, we first parametrize $M$. In cylindrical coordinates, we define

$$
\psi(\theta, z) = (\cos \theta, \sin \theta, z),
$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 2$. The tangent vectors are

$$
\frac{\partial \psi}{\partial \theta} = (-\sin \theta, \cos \theta, 0)
$$

and

$$
\frac{\partial \psi}{\partial z} = (0, 0, 1).
$$

The normal vector is

$$
\frac{\partial \psi}{\partial \theta} \times \frac{\partial \psi}{\partial z} = (\cos \theta, \sin \theta, 0).
$$

Since $X(\psi) = (\cos^3 \theta, \sin^3 \theta, z^3)$, we have

$$
\left( X(\psi), \frac{\partial \psi}{\partial \theta} \times \frac{\partial \psi}{\partial z} \right) = \cos^4 \theta + \sin^4 \theta.
$$

Prior to integrating, let’s apply some trigonometric identities, namely

$$
\sin(2\theta) = 2 \sin \theta \cos \theta
$$

and

$$
\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos(2\theta).
$$
Then,

\[
\cos^4 \theta + \sin^4 \theta = \cos^2 \theta (1 - \sin^2 \theta) + \sin^2 \theta (1 - \cos^2 \theta) \\
= 1 - 2 \sin^2 \theta \cos^2 \theta \\
= 1 - 2 \left( \frac{1}{2} \sin(2\theta) \right)^2 \\
= 1 - \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2} \cos(4\theta) \right) \\
= \frac{3}{4} + \frac{1}{4} \cos(4\theta).
\]

Hence, the flux is

\[
\int_0^2 \int_0^{2\pi} \left( \mathbf{X}(\psi), \frac{\partial \psi}{\partial \theta} \times \frac{\partial \psi}{\partial z} \right) d\theta dz = \int_0^2 \int_0^{2\pi} \left( \frac{3}{4} + \frac{1}{4} \cos(4\theta) \right) d\theta dz \\
= \int_0^2 \left( \frac{3\pi}{4} + \frac{1}{16} \sin(4\theta) \right) \left|_0^{2\pi} \right. dz \\
= \int_0^2 \frac{3\pi}{2} dz \\
= 3\pi.
\]

In the end, the divergence theorem makes short work of finding the answer, however, it must be used carefully. We want to compute the flux of \( \mathbf{X} \) across the cylinder \( M \). In order to apply the divergence theorem, we need a 3-surface, call it \( B \), such that the boundary of \( B \) is \( M \). In this case, we let \( B \) be the solid cylinder (or rod). Rather than \( x^2 + y^2 = 1 \), for \( B \) we assume \( x^2 + y^2 \leq 1 \). Using cylindrical coordinates, we can parametrize the 3-surface \( B \) by

\[
F(r, \theta, z) = (r \cos \theta, r \sin \theta, z),
\]

where \( 0 \leq r \leq 1 \), \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq z \leq 2 \). Before proceeding with the calculation, there is a minor issue to address with this approach. The boundary of \( B \) consists of 3 pieces, only one of which is \( M \). The boundary of \( B \) consists of two faces (top and bottom), in addition to the cylindrical boundary \( M \) (similar to Figure 4.4 in the course notes). Applying the divergence theorem will calculate the flux of \( \mathbf{X} \) across all portions of the boundary. However, we can adjust our answer.

Now, the divergence of \( \mathbf{X} \) is

\[
\text{div} \ X = 3x^2 + 3y^2 + 3z^2,
\]

which is \( 3r^2 + 3z^2 \) in terms of our chosen coordinate system. The last piece of information we need, before integrating, is the volume form for \( B \), \( dV_B \). To obtain the
volume form for $B$, we compute the pullback of $dx \wedge dy \wedge dz$ by $F$. We have

\[
F^*(dx \wedge dy \wedge dz) = d(r \cos \theta) \wedge d(r \sin \theta) \wedge dz
\]

\[
= (\cos \theta dr - r \sin \theta d\theta) \wedge (\sin \theta dr + r \cos \theta d\theta) \wedge dz
\]

\[
= (r \cos^2 \theta + r \sin^2 \theta) dr \wedge d\theta \wedge dz
\]

\[
= rdr \wedge d\theta \wedge dz.
\]

Thus, we integrate

\[
\int_0^{2\pi} \int_0^2 \int_0^1 3(r^2 + z^2) rdr dz d\theta = \int_0^{2\pi} \int_0^2 \left( \frac{3}{4} r^4 + \frac{3}{2} r^2 z^2 \right) \bigg|_0^1 \, dz 
\]

\[
= \int_0^{2\pi} \left( \frac{3}{4} z^2 + \frac{1}{2} z^3 \right) \bigg|_0^2 \, d\theta
\]

\[
= \int_0^{2\pi} \left( \frac{3}{2} + 4 \right) \, d\theta
\]

\[
= 11\pi.
\]

This is the flux through all three boundary portions. In order to find just the flux across $M$, we must subtract the flux through the top and bottom boundary faces. These values can be obtained by direct calculation.

The top corresponds to the conditions $z = 2, x^2 + y^2 \leq 1$. We parametrize the top by

\[
\psi(r, \theta) = (r \cos \theta, r \sin \theta, 2),
\]

where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. The tangent vectors are

\[
\frac{\partial \psi}{\partial r} = (\cos \theta, \sin \theta, 0)
\]

and

\[
\frac{\partial \psi}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0).
\]

The normal vector is

\[
\frac{\partial \psi}{\partial r} \times \frac{\partial \psi}{\partial \theta} = (0, 0, r).
\]

We evaluate the vector field $X$ at $\psi$ to obtain

\[
X(\psi) = (r^3 \cos^3 \theta, r^3 \sin^3 \theta, 8).
\]

The flux through the top is

\[
\int_0^{2\pi} \int_0^1 \left( X(\psi), \frac{\partial \psi}{\partial r} \times \frac{\partial \psi}{\partial \theta} \right) \, dr \, d\theta = \int_0^{2\pi} \int_0^1 8r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} 4 \, d\theta
\]

\[
= 8\pi.
\]
Similarly, the \textit{bottom} corresponds to the conditions \( z = 0, \ x^2 + y^2 \leq 1 \). We parametrize the bottom by

\[
\psi(r, \theta) = (r \cos \theta, r \sin \theta, 0),
\]

where \( 0 \leq r \leq 1 \) and \( 0 \leq \theta \leq 2\pi \). The tangent vectors are

\[
\frac{\partial \psi}{\partial r} = (\cos \theta, \sin \theta, 0)
\]

and

\[
\frac{\partial \psi}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0).
\]

The normal vector is

\[
\frac{\partial \psi}{\partial r} \times \frac{\partial \psi}{\partial \theta} = (0, 0, r).
\]

We evaluate the vector field \( X \) at \( \psi \) to obtain

\[
X(\psi) = (r^3 \cos^3 \theta, r^3 \sin^3 \theta, 0).
\]

However, this time we have

\[
\left\langle X(\psi), \frac{\partial \psi}{\partial r} \times \frac{\partial \psi}{\partial \theta} \right\rangle = 0,
\]

hence the flux through the bottom is 0.

Finally, subtracting the flux through the top and bottom from \( 11\pi \), we obtain the flux across \( M \), which is \( 3\pi \).

\textbf{P3.} Let \( X = (x^2yz, yz^2, z^3e^{xy}) \). Determine the flux of \( \text{curl} \ X \) across the 2-surface \( M \), where \( M \) is the part of the sphere \( x^2 + y^2 + z^2 = 5 \) above \( z = 1 \), oriented by the upward normal.

\textbf{Solution 3.} \textit{Answer:} \(-4\pi\)

By classical Stokes’ theorem, we need only compute the line integral of \( X \) over the boundary of \( M \). The boundary of \( M \) occurs for \( z = 1 \), at which \( x^2 + y^2 = 4 \). Since \( M \) is oriented by the upward normal, the boundary is a circle of radius 2 oriented counterclockwise (viewed from above), with \( z = 1 \). We parametrize \( \partial M \) by

\[
\gamma(t) = (2 \cos t, 2 \sin t, 1), \quad \text{for } 0 \leq t \leq 2\pi.
\]

The tangent vector is \( \gamma'(t) = (-2 \sin t, 2 \cos t, 0) \) and we have

\[
\int_0^{2\pi} \langle X(\gamma), \gamma' \rangle \, dt = \int_0^{2\pi} (-16 \sin^2 t \cos^2 t + 4 \sin t \cos t) \, dt = -4\pi.
\]
P4. Let \( X = (xz, -2y, 3x) \). Determine the flux of \( X \) across the sphere of radius 2, oriented by the exterior normal.

**Solution 4. Answer:** \(-\frac{64\pi}{3}\)

Here, applying the divergence theorem is straightforward. The sphere of radius 2 is the boundary of the ball of radius 2. Parametrizing the ball by

\[
F(\rho, \varphi, \theta) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi),
\]

with \( 0 \leq \rho \leq 2, 0 \leq \varphi \leq \pi \) and \( 0 \leq \theta \leq 2\pi \), the corresponding volume form is

\[
dV_B = \rho^2 \sin \varphi \; d\rho \wedge d\varphi \wedge d\theta.
\]

We have \( \text{div} \; X = z - 2 \), hence we integrate

\[
\int_0^{2\pi} \int_0^{\pi} \int_0^2 (\rho \cos \varphi - 2) \rho^2 \sin \varphi \; d\rho \; d\varphi \; d\theta = \int_0^{2\pi} \int_0^{\pi} \left( \frac{1}{4} \rho^4 \sin \varphi \cos \theta - \frac{2}{3} \rho^3 \sin \varphi \right) \Big|_0^2 d\varphi \; d\theta
\]

\[
= \int_0^{2\pi} \left( 2 \sin^2 \varphi + \frac{16}{3} \cos \varphi \right) \Big|_0^\pi d\theta
\]

\[
= \int_0^{2\pi} \frac{32}{3} d\theta
\]

\[
= -\frac{64\pi}{3}.
\]

P5. Compute the line integral of the covector field \( \omega \) over the parametrized curve \( \gamma \).

a) \( \omega = e^y \; dx + (xe^y + e^z) \; dy + ye^z \; dz \) and \( \gamma \) is any path that connects \((0, 2, 0)\) to \((4, 0, 3)\).

b) \( \omega = (4x^3y^2 - 2xy^3) \; dx + (2x^4y - 3x^2y^2 + 4y^3) \; dy \) and \( \gamma(t) = (t + \sin(\pi t), 2t + \cos(\pi t)) \), \( 0 \leq t \leq 1 \).

**Solution 5.**

**Solution 5a) Answer:** 2

The function \( f(x, y, z) = xe^y + ye^z \) is a potential for \( \omega \). By the fundamental theorem of line integrals,

\[
\int_\gamma \omega = f(4, 0, 3) - f(0, 2, 0)
\]

\[
= (4e^0 + 0) - (0 + 2e^0)
\]

\[
= 2.
\]
Solution 5b) Answer: 0

The function \( f(x, y) = x^4y^2 - x^2y^3 + y^4 \) is a potential for \( \omega \). By the fundamental theorem of line integrals,

\[
\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))
= f(1,1) - f(0,1)
= (1 - 1 + 1) - (1)
= 0.
\]

P6. Consider the gravitational vector field

\[
X = G \frac{Mm}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} (x, y, z),
\]

where \( m, M \) and \( G \) are constants. Determine the flux of \( X \) across the sphere of radius \( a > 0 \).

Solution 6.
Answer: \( \pm 4\pi GMm \) Hint: The divergence theorem does not apply, as one might expect, so evaluate directly. Note that, if such a problem were to be on the final exam, you would be told that the divergence theorem can not be used.

P7. Let \( \omega = e^{-x} \, dx + e^x \, dy + e^z \, dz \) and let \( M \) be the portion of the plane \( 2x + y + 2z = 2 \) in the first octant. Verify the general Stokes’ theorem

\[
\int_{\partial M} \omega = \int_M d\omega.
\]

Solution 7. Answer: \( 2e - 4 \)

(Left-hand side)

If we assume an upward pointing normal for \( M \), the boundary of \( M \) consists of line segments from \((1, 0, 0)\) to \((0, 2, 0)\) to \((0, 0, 1)\) to \((1, 0, 0)\). The segment from \((1, 0, 0)\) to \((0, 2, 0)\) can be parametrized by \( \gamma_1(t) = (1 - t, 2t, 0) \), where \( 0 \leq t \leq 1 \). The segment from \((0, 2, 0)\) to \((0, 0, 1)\) can be parametrized by \( \gamma_2(t) = (0, 2 - 2t, t) \), where \( 0 \leq t \leq 1 \). The
segment from (0, 0, 1) to (1, 0, 0) can be parametrized by $\gamma_3(t) = (t, 0, 1 - t)$, where $0 \leq t \leq 1$. We compute the pullback of $\omega$ by $\gamma_1$, $\gamma_2$ and $\gamma_3$,

$$
\gamma_1^* \omega = (-e^{t-1} + 2e^{1-t}) \, dt \quad \gamma_2^* \omega = (2 + e^t) \, dt \quad \gamma_3^* \omega = (e^{-t} - e^{1-t}) \, dt.
$$

Thus,

$$\int_{\partial M} \omega = \int_0^1 \gamma_1^* \omega + \gamma_2^* \omega + \gamma_3^* \omega$$

$$= \int_0^1 (e^{1-t} - e^{t-1} + e^t + e^{-t} - 2) \, dt$$

$$= (e^{1-t} - e^{t-1} + e^t - e^{-t} - 2) \bigg|_0^1$$

$$= 2e - 4.
$$

(Right-hand side)

The exterior derivative of $\omega$ is

$$d\omega = -e^{-x} dx \wedge dx + e^x dx \wedge dy + e^z dz \wedge dz = e^x dx \wedge dy.$$

The 2-surface $M$ can be parametrized by

$$F(u, v) = (u, v, 1 - \frac{1}{2}v - u),$$

where $0 \leq u \leq 1$ and $0 \leq v \leq 2 - 2u$. To compute the integral, we first pullback $d\omega$ by $F$,

$$F^*(d\omega) = e^u \, du \wedge dv,$$

which, in this case, is trivial since $u, v$ are substitutes for $x, y$. Thus,

$$\int_M d\omega = \int_0^1 \int_0^{2-2u} e^u \, dv \, du$$

$$= \int_0^1 2e^u - 2ue^u \, du$$

$$= (2e^u - 2ue^u + 2e^u) \bigg|_0^1$$

$$= 2e - 4.$$
P8. Let \( \omega = 2z\,dx + 4x\,dy + 5y\,dz \) and let \( M \) be the portion of the cylinder \( x^2 + y^2 = 4 \) bounded by the \( xy \)-plane and the plane \( z = x + 4 \). Verify the general Stokes’ theorem
\[
\int_{\partial M} \omega = \int_M d\omega.
\]

**Solution 8.** Answer: \(-20\pi\)

*(Left-hand side)*

The surface \( M \) has two boundary pieces, one at the top and one at the bottom. The top corresponds to the intersection of the cylinder with the plane \( z = x + 4 \). Substituting \( x = z - 4 \) into \( x^2 + y^2 = 4 \), we obtain \((z - 4)^2 + y^2 = 4\). The last equation represents a circle of radius 2, centered at \((0, 4)\) in the \( yz \)-plane. This suggests that we take \( y = 2\sin t \) and \( z = 4 + 2\cos t \). Since \( x = z - 4 \), we also have \( x = 2\cos t \). Thus, the top boundary can be parametrized by
\[
\gamma(t) = (2\cos t, 2\sin t, 4 + 2\cos t),
\]
with \( 0 \leq t \leq 2\pi \). Note that we have oriented the top boundary counterclockwise, when viewed from above. Computing the pullback of \( \omega \) by \( \gamma \), we have
\[
\gamma^* \omega = 2(2 + 2\cos t)(-2\sin t)\,dt + 4(2\cos t)2\cos t\,dt + 5(2\sin t)(-2\sin t)\,dt
\]
\[
= (16\sin t - 8\sin t\cos t + 16\cos^2 t - 20\sin^2 t)\,dt.
\]

Hence, we integrate
\[
\int_0^{2\pi} (16\sin t - 8\sin t\cos t + 16\cos^2 t - 20\sin^2 t)\,dt = \left[ 16\cos t + 4\cos^2 t - 2t + 9\sin 2t \right]_0^{2\pi}
\]
\[
= -4\pi.
\]

The bottom boundary is just a circle of radius 2 in the \( xy \)-plane, which we can parametrize by \( \xi(t) = (2\cos t, -2\sin t, 0) \), where \( 0 \leq t \leq 2\pi \). Note that we have oriented the bottom boundary counterclockwise when viewed from below (why?). The pullback of \( \omega \) by \( \xi \) is
\[
\xi^* \omega = 4(2\cos t)(-2\cos t)\,dt = -16\cos^2 t\,dt.
\]
Hence, we integrate

\[
\int_0^{2\pi} -16 \cos^2 t \, dt = -8 \int_0^{2\pi} (1 + \cos 2t) \, dt \\
= (-8t - 4 \sin 2t) \bigg|_0^{2\pi} \\
= -16\pi.
\]

Summing the two boundary contributions yields \(-20\pi\).

(Right-hand side)

The surface \(M\) can be parametrized in cylindrical coordinates by

\[
F(\theta, z) = (2\cos \theta, 2\sin \theta, z),
\]

where \(0 \leq \theta \leq 2\pi\) and \(0 \leq z \leq 2\cos \theta + 4\). The exterior derivative of \(\omega\) is

\[
d\omega = 2 \, dz \wedge dx + 4 \, dx \wedge dy + 5 \, dy \wedge dz.
\]

The pullback of \(d\omega\) by \(F\) is

\[
F^*(d\omega) = 2 \, dz \wedge d(2\cos \theta) + 4 \, d(2\cos \theta) \wedge d(2\sin \theta) + 5 \, d(2\cos \theta) \wedge d\theta \\
= -4\sin \theta \, dz \wedge d\theta + 10 \cos \theta \, d\theta \wedge dz \\
= (-4\sin \theta - 10\cos \theta) \, dz \wedge d\theta.
\]

Hence,

\[
\int_M d\omega = \int_0^{2\pi} \int_0^{2\cos \theta + 4} (-4\sin \theta - 10\cos \theta) \, dz \, d\theta \\
= \int_0^{2\pi} (2\cos \theta + 4)(-4\sin \theta - 10\cos \theta) \, d\theta \\
= \int_0^{2\pi} (-8\sin \theta \cos \theta - 16\sin \theta - 20\cos^2 \theta - 40\cos \theta) \, d\theta \\
= \left(4\cos^2 \theta + 16\cos \theta - 10\theta - 5\sin(2\theta) - 40\sin \theta \right) \bigg|_0^{2\pi} \\
= -20\pi.
\]

P9. Let \(\omega = x \, dx + y \, dy + (x^2 + y^2) \, dz\) and let \(M\) be the portion of the paraboloid \(z = 1 - x^2 - y^2\) in the first octant. Verify the general Stokes’ theorem

\[
\int_{\partial M} \omega = \int_M d\omega.
\]
Solution 9. Answer: 0

(Left-hand side)

The boundary, \( \partial M \), consists of three pieces corresponding to the intersection of the paraboloid with the three coordinate planes. Let's assume the boundary is oriented counterclockwise, as viewed from the first octant. When \( y = 0 \), the paraboloid intersects the \( xz \)-plane, and we have \( z = 1 - x^2 \). We can parametrize this portion by \( \gamma_1(t) = (t, 0, 1 - t^2) \), for \( 0 \leq t \leq 1 \). When \( z = 0 \), the paraboloid intersects the \( xy \)-plane, and we have \( x^2 + y^2 = 1 \). We can parametrize this portion by \( \gamma_2(t) = (\cos t, \sin t, 0) \), for \( 0 \leq t \leq \pi/2 \). When \( x = 0 \), the paraboloid intersects the \( yz \)-plane, and we have \( z = 1 - y^2 \). We can parametrize this portion by \( \gamma_3(t) = (0, t, 1 - t^2) \), for \( 0 \leq t \leq 1 \). Note that \( \gamma_1 \) is a curve from \((0, 0, 1)\) to \((1, 0, 0)\), \( \gamma_2 \) is a curve from \((1, 0, 0)\) to \((0, 1, 0)\), and \( \gamma_3 \) is a curve from \((0, 0, 1)\) to \((0, 1, 0)\). The parametrization \( \gamma_3 \) is in the opposing direction of rotation of \( \gamma_1 \) and \( \gamma_2 \). Thus, we must either redefine \( \gamma_3 \) with the correct orientation or we could negate the integral of \( \omega \) over \( \gamma_3 \). A suitable definition of \( \gamma_3 \) would be \( \gamma_3(t) = (0, 1 - t, 2t - t^2) \), for \( 0 \leq t \leq 1 \).

Now, the corresponding pullbacks are
\[
\begin{align*}
\gamma_1^* \omega &= t \, dt + t^2(-2t) \, dt = (t - 2t^3) \, dt \\
\gamma_2^* \omega &= \cos t(-\sin t) \, dt + \sin t(\cos t) \, dt = 0 \\
\gamma_3^* \omega &= -(1 - t) \, dt + (1 - t)^2(2 - 2t) \, dt = (-2t^3 + 6t^2 - 5t + 1) \, dt.
\end{align*}
\]

Thus, we integrate
\[
\int_{\partial M} \omega = \int_0^1 \gamma_1^* \omega + \int_0^{2\pi} \gamma_2^* \omega + \int_0^1 \gamma_3^* \omega \\
= \int_0^1 (-4t^3 + 6t^2 - 4t + 1) \, dt \\
= (-t^4 + 2t^3 - 2t^2 + t)|_0^1 \\
= 0.
\]

(Right-hand side)

Using cylindrical coordinates, we parametrize \( M \) by
\[
F(r, \theta) = (r \cos \theta, r \sin \theta, 1 - r^2),
\]
where \( 0 \leq r \leq 1 \) and \( 0 \leq \theta \leq \pi/2 \). The exterior derivative of \( \omega \) is
\[
d\omega = 2x \, dx \wedge dz + 2y \, dy \wedge dz.
\]
The pullback of $d\omega$ by $F$ is
\begin{align*}
F^*(d\omega) &= 2r \cos \theta \, d(r \cos \theta) \wedge d(1 - r^2) + 2r \sin \theta \, d(r \sin \theta) \wedge d(1 - r^2) \\
&= 2r \cos \theta (-r \sin \theta)(-2r) \, d\theta \wedge dr + 2r \sin \theta (r \cos \theta)(-2r) \, d\theta \wedge dr \\
&= (4r^3 \sin \theta \cos \theta - 4r^3 \sin \theta \cos \theta) \, d\theta \wedge dr \\
&= 0.
\end{align*}

Since the pullback is 0, there is no need to integrate.

**P10.** Suppose the position of a particle at time $t$ is given by the parametrized curve $\gamma(t) = (e^t \cos t, e^t \sin t)$. If $X = (x, y)$, determine the work done by the vector force field $X$ on the particle as it traverses from $\gamma(0)$ to $\gamma(\pi/2)$.

**Solution 10.** Answer: $\frac{1}{2}(e^\pi - 1)$

*(Direct Method)*

To compute the work, we need the line integral of $X$ over $\gamma$

$$\int_{\gamma} X = \int_0^{\pi/2} \langle X(\gamma), \gamma' \rangle \, dt.$$  

The vector field $X$ evaluated on $\gamma$ is $X(\gamma) = (e^t \cos t, e^t \sin t)$ and the tangent (velocity) vector is $\gamma'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t)$. The inner product is

$$\langle X(\gamma), \gamma' \rangle = e^t \cos t (e^t \cos t - e^t \sin t) + e^t \sin t (e^t \sin t + e^t \cos t)$$

$$= e^{2t} (\cos^2 t - \cos t \sin t + \sin^2 t + \cos t \sin t)$$

$$= e^{2t} (\cos^2 t + \sin^2 t)$$

$$= e^{2t}.$$

So, the work is given by

$$\int_0^{\pi/2} \langle X(\gamma), \gamma' \rangle \, dt = \int_0^{\pi/2} e^{2t} \, dt$$

$$= \frac{1}{2} e^{2t} \bigg|_0^{\pi/2}$$

$$= \frac{1}{2} (e^\pi - 1).$$

*(Fundamental Theorem of Line Integrals)*
We could have obtained the solution much quicker by using the Fundamental Theorem of Line Integrals. It is straightforward to check that $f(x, y) = \frac{1}{2}(x^2 + y^2)$ is a potential for $X$, that is $\text{grad} f = X$. By the fundamental theorem,

$$\int_{0}^{\pi/2} \langle X(\gamma), \gamma' \rangle \, dt = f(\gamma(\pi/2)) - f(\gamma(0)).$$

Now,

$$\gamma(\pi/2) = (0, e^{\pi/2})$$

and

$$\gamma(0) = (1, 0),$$

hence

$$f(\gamma(\pi/2)) - f(\gamma(0)) = \frac{1}{2}(0 + (e^{\pi/2})^2) - \frac{1}{2}(1^2 + 0)$$

$$= \frac{1}{2}(e^\pi - 1).$$

**P11.** Determine the area of the following surfaces.

a) The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

b) The part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$, where $a > 0$.

c) The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$.

d) The part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$.

e) The part of the paraboloid $z = x^2 + y^2$ between $z = 0$ and $z = 1$, including the top (the intersection of $z = 0$ and $z = x^2 + y^2$).

f) The surface described by $z = xy$, where $x^2 + y^2 \leq 2$.

**Solution 11.**

**Solution 11a)** Answer: $4\pi$
Before parametrizing, let’s see where the sphere and paraboloid intersect. They intersect when both
\[ x^2 + y^2 + z^2 = 4z \]
and
\[ z = x^2 + y^2 \]
are satisfied. Hence,
\[ z + z^2 = 4z \]
and solving for \( z \) yields two solutions, \( z = 0 \) and \( z = 3 \). It may take a little bit of thought, but the portion of the sphere inside the paraboloid is \( z \geq 3 \). Any point on the sphere with \( 0 \leq z < 3 \) would be outside the paraboloid. Thus, whichever coordinate system we use, we want the corresponding \( z \) coordinate to satisfy \( z \geq 3 \).

\textbf{(Spherical coordinates)}

In general, the equation of a sphere of radius \( a \) centered at the point \((h, k, \ell)\) is
\[ (x - h)^2 + (y - k)^2 + (z - \ell)^2 = a^2. \]
The corresponding parametrization in spherical coordinates would be
\[ F(\varphi, \theta) = (h + a \sin \varphi \cos \theta, k + a \sin \varphi \sin \theta, \ell + a \cos \varphi). \]
In this case, if we complete the square with respect to \( z \), the equation of the sphere can be rewritten
\[ x^2 + y^2 + (z - 2)^2 = 4. \]
So, the sphere is centered at \((0, 0, 2)\) and has radius 2. Thus, we can parametrize the sphere with
\[ F(\varphi, \theta) = (2 \sin \varphi \cos \theta, 2 \sin \varphi \sin \theta, 2 + 2 \cos \varphi). \]
Then, the condition \( z \geq 3 \) becomes \( 2 + 2 \cos \varphi \geq 3 \), hence
\[ \cos \varphi \geq \frac{1}{2}, \]
which is satisfied for \( 0 \leq \varphi \leq \frac{\pi}{3} \). The values of \( \theta \) are within the standard range \( 0 \leq \theta \leq 2\pi \). With this parametrization, we have tangent vectors
\[ \frac{\partial F}{\partial \varphi} = (2 \cos \varphi \cos \theta, 2 \cos \varphi \sin \theta, -2 \sin \varphi) \]
and
\[ \frac{\partial F}{\partial \theta} = (-2 \sin \varphi \sin \theta, 2 \sin \varphi \cos \theta, 0). \]
The corresponding normal vector is
\[ \frac{\partial F}{\partial \varphi} \times \frac{\partial F}{\partial \theta} = (4\sin^2\varphi \cos \theta, 4\sin^2\varphi \sin \theta, 4\sin \varphi \cos \varphi) \]
which has norm
\[ \left\| \frac{\partial F}{\partial \varphi} \times \frac{\partial F}{\partial \theta} \right\| = \left( 16\sin^4 \varphi \cos^2 \theta + 16\sin^4 \varphi \sin^2 \theta + 16\sin^2 \varphi \cos^2 \varphi \right)^{\frac{1}{2}} \]
\[ = \left( 16\sin^4 \varphi + 16\sin^2 \varphi \cos^2 \varphi \right)^{\frac{1}{2}} \]
\[ = \left( 16\sin^2 \varphi \right)^{\frac{1}{2}} \]
\[ = 4\sin \varphi. \]
Thus, the surface area is
\[ \int_0^{2\pi} \int_0^{\pi/3} 4\sin \varphi \, d\varphi \, d\theta = \int_0^{2\pi} -4\cos \theta \bigg|_0^{\pi/3} \, d\theta \]
\[ = \int_0^{2\pi} -4 \left( \frac{1}{2} - 1 \right) \, d\theta \]
\[ = 2 \int_0^{2\pi} \, d\theta \]
\[ = 4\pi. \]
Note that the spherical parametrization
\[ F(\varphi, \theta) = (4\cos \varphi \sin \varphi \cos \theta, 4\cos \varphi \sin \varphi \sin \theta, 4\cos^2 \varphi) \]
is also valid, but the bounds for \( \varphi \) are given by \( \frac{5\pi}{6} \leq \varphi \leq \pi \).

(Cylindrical coordinates)
In cylindrical coordinates, the equation for the sphere becomes
\[ r^2 + z^2 = 4z \]
and the equation for the paraboloid becomes
\[ z = r^2. \]
The intersection can be found by solving
\[ r^2 + r^4 = 4r^2 \]
for $r$, which yields $r = 0$ and $r = \sqrt{3}$, so $0 \leq r \leq \sqrt{3}$. Note that, since the sphere is centered at $(0, 0, 2)$ and the radius is 2, we have $z \geq 0$. Solving the equation $r^2 + z^2 = 4z$ yields $z = 2 + \sqrt{4 - r^2}$, but since $z \geq 0$ we only have $z = 2 + \sqrt{4 - r^2}$. Thus, we can parametrize the sphere by

$$F(r, \theta) = (r \cos \theta, r \sin \theta, 2 + \sqrt{4 - r^2}),$$

with $0 \leq r \leq \sqrt{3}$ and $0 \leq \theta \leq 2\pi$. In this case, the norm of the normal vector is

$$\left\| \frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} \right\| = \frac{2r}{\sqrt{4 - r^2}}.$$

Thus, the surface area is

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}} \ dr \ d\theta = \int_0^{2\pi} \left[ -2\sqrt{4 - r^2} \right]_0^{\sqrt{3}} \ d\theta = \int_0^{2\pi} -2(1 - 2) \ d\theta = 2 \int_0^{2\pi} \ d\theta = 4\pi.$$

**Solution 11b)**  
Answer: $2a^2(\pi - 2)$

**Solution 11c)**  
Answer: $\frac{\sqrt{2}}{6}$

Parametrize the paraboloid by

$$F(r, \theta) = (r \cos \theta, r \sin \theta, r).$$

Then, the condition $y = x$ becomes $r \sin \theta = r \cos \theta$, which implies $r = 0$ or $\theta = n/4$. The condition $y = x^2$ becomes $r \sin \theta = r^2 \cos^2 \theta$, which implies $r = 0$ or

$$r = \frac{\sin \theta}{\cos^2 \theta} = \tan \theta \sec \theta.$$

The bounds for $r, \theta$ are $0 \leq \theta \leq n/4$ and $0 \leq r \leq \tan \theta \sec \theta$. The tangent vectors are

$$\frac{\partial F}{\partial r} = (\cos \theta, \sin \theta, 1)$$

and

$$\frac{\partial F}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0).$$
The normal vector is
\[ \frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} = (-r \cos \theta, -r \sin \theta, r), \]
which has norm
\[ \left\| \frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} \right\| = \sqrt{2} r. \]

Hence, we integrate
\[ \int_0^{\pi/4} \int_0^{\tan \theta \sec \theta} \sqrt{2} r \, dr \, d\theta = \int_0^{\pi/4} \frac{\sqrt{2}}{2} \tan^2 \theta \sec^2 \theta \, d\theta = \frac{\sqrt{2}}{6} \tan^3 \theta \bigg|_0^{\pi/4} = \frac{\sqrt{2}}{6}. \]

Note that the last integration step used \( u \)-substitution with \( u = \tan \theta \).

**Solution**

**d)** Answer: \( \frac{\pi}{6} \left( 65^{3/2} - 1 \right) \)

The paraboloid can be parametrized by
\[ F(r, \theta) = (r \cos \theta, r^2, r \sin \theta). \]
The portion inside the cylinder satisfies \( x^2 + z^2 \leq 16 \), hence \( r^2 \leq 16 \). So, the bounds are \( 0 \leq r \leq 4 \) and \( 0 \leq \theta \leq 2\pi \). The tangent vectors are
\[ \frac{\partial F}{\partial r} = (\cos \theta, 2r, \sin \theta) \]
and
\[ \frac{\partial F}{\partial \theta} = (-r \sin \theta, 0, r \cos \theta). \]
The normal vector is
\[ \frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} = (2r^2 \cos \theta, -r, 2r^2 \sin \theta), \]
which has norm
\[ \left\| \frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} \right\| = \sqrt{4r^4 + r^2} = r \sqrt{4r^2 + 1}. \]

We integrate
\[ \int_0^{2\pi} \int_0^4 r \sqrt{4r^2 + 1} \, dr \, d\theta = \int_0^{2\pi} \frac{1}{12} (4r^2 + 1)^{3/2} \bigg|_0^4 \, d\theta = \int_0^{2\pi} \frac{1}{12} (65^{3/2} - 1) \, d\theta = \frac{\pi}{6} (65^{3/2} - 1). \]
Solution 11e) Answer: $\frac{\pi}{6} \left(5\sqrt{5} + 5\right)$ or $\frac{\pi}{6} \left(5\sqrt{5} - 1\right)$ if the top is excluded.

Excluding the top, this problem is nearly identical to the previous one. The paraboloid can be parametrized by

$$F(r, \theta) = (r \cos \theta, r \sin \theta, r^2).$$

The bounds are $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. The tangent vectors are

$$\frac{\partial F}{\partial r} = (\cos \theta, \sin \theta, 2r)$$

and

$$\frac{\partial F}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0).$$

The normal vector is

$$\frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} = (-2r^2 \cos \theta, -2r^2 \sin \theta, r),$$

which has norm

$$\left\| \frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} \right\| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1}.$$

We integrate

$$\int_0^{2\pi} \int_0^1 r\sqrt{4r^2 + 1} \, dr \, d\theta = \int_0^{2\pi} \left. \frac{1}{12} (4r^2 + 1)^{3/2} \right|_0^1 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} (5^{3/2} - 1) \, d\theta$$

$$= \frac{\pi}{6} (5^{3/2} - 1).$$

The top is a disc at $z = 1$, hence $x^2 + y^2 = 1$. So, the top is simply a circle of radius 1, which has area $\pi$. Adding $\pi$ to the previous surface area and combining terms yields

$$\frac{\pi}{6} (5^{3/2} + 5).$$

Solution 11f) Answer: $\frac{2\pi}{3} \left(3^{3/2} - 1\right)$

P12. Let $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Compute

$$\int_{\gamma} -y \, dx + x \, dy.$$
Solution 12. 
Answer: $2\pi$

\textbf{P13.} Let $\gamma(t)$ be a parametrization of the path from $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$. Compute 
\[
\int_{\gamma} yz \, dx + xz \, dy + xy \, dz.
\]

\textbf{Solution 13. Answer: 0}
Let $\omega = yz \, dx + xz \, dy + xy \, dz$. The covector field $\omega$ is exact, with potential function $f(x, y, z) = xyz$. In fact, the vector field $X$ of P15 corresponds to $\omega$ ($\omega^\flat = X$). By the fundamental theorem of line integrals, 
\[
\int_{\gamma} \omega = f(0, 0, 1) - f(1, 0, 0)
= 0.
\]
Note that, there is no need to consider the point $(0, 1, 0)$. Since $\omega$ is exact, only the initial and final point of the path matter.
P14. Let \( X = \left( \frac{y^2}{x^2}, \frac{-2y}{x} \right) \).

a) Is \( X \) a conservative vector field? If so, determine a potential for \( X \).

b) Compute the work done by \( X \) in moving a particle from \((1,1)\) to \((4,-2)\).

\textbf{Solution 14.}

\textbf{Solution 14a)} \textit{Answer:} yes, \( f(x,y) = \frac{-y^2}{x} \)

If \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( \frac{y^2}{x^2}, \frac{-2y}{x} \right) \), then

\[ \frac{\partial f}{\partial x} = \frac{y^2}{x^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{-2y}{x} . \]

Integrating the first equation with respect to \( x \) yields

\[ -\frac{y^2}{x} \]

and integrating the second equation with respect to \( y \) yields

\[ -\frac{y^2}{x} . \]

Thus,

\[ f(x,y) = -\frac{y^2}{x} \]

is a potential for \( X \), which can be verified by computing

\[ \text{grad } f = \left( \frac{y^2}{x^2}, \frac{-2y}{x} \right) . \]

Since \( X \) has a potential, it is a conservative vector field.

\textbf{Solution 14b)} \textit{Answer:} 0

Since \( X \) is conservative, we do not need a path to integrate over, we just need a starting and ending point. The work done in moving a particle from \((1,1)\) to \((4,-2)\) is

\[ f(4,-2) - f(1,1) = -\frac{(-2)^2}{4} - \left( -\frac{1^2}{1} \right) \]

\[ = -1 + 1 \]

\[ = 0. \]
We can verify that if \( \gamma \) is any path such that \( \gamma(a) = (1, 1) \) and \( \gamma(b) = (4, -2) \), then
\[
\int_a^b \langle X(\gamma), \gamma' \rangle \, dt = 0.
\]
For example, try the straight line segment
\[
\gamma(t) = (1 + 3t, 1 - 3t)
\]
for \( 0 \leq t \leq 1 \).

**P15.** Let \( X = (yz, xz, xy) \) be a vector field on \( \mathbb{R}^3 \) and let \( \gamma(t) = (\cos t, \sin t, t) \), \( 0 \leq t \leq \pi/4 \), be a parametrized curve.

a) Compute the line integral
\[
\int_{\gamma} X.
\]

b) Show that \( X \) is conservative and use the Fundamental Theorem of Line Integrals to compute
\[
\int_{\gamma} X.
\]

**Solution 15.**

**Solution 15a) Answer: \( \frac{\pi}{8} \)**

We have
\[
X(\gamma) = (t \sin t, t \cos t, \sin t \cos t)
\]
and
\[
\gamma'(t) = (-\sin t, \cos t, 1),
\]
so
\[
\langle X(\gamma), \gamma' \rangle = t(\cos^2 t - \sin^2 t) + \sin t \cos t = t \cos(2t) + \sin t \cos t.
\]
Thus,

\[ \int_X = \int_0^{\pi/4} \langle X(\gamma), \gamma' \rangle \, dt \]

\[ = \int_0^{\pi/4} t \cos(2t) + \sin t \cos t \, dt \]

\[ = \left( \frac{t}{2} \sin(2t) + \frac{1}{4} \cos(2t) + \frac{1}{2} \sin^2 t \right) \bigg|_0^{\pi/4} \]

\[ = \frac{\pi}{8} \]

**Solution 15 b)** Answer: \( \frac{\pi}{8} \), \( f(x, y, z) = xyz \) is a potential.

The function \( f(x, y, z) = xyz \) is a potential for \( X \), since

\[ \text{grad} \, f = (yz, xz, xy) = X. \]

By the fundamental theorem of line integrals

\[ \int_X = f(\gamma(\pi/4)) - f(\gamma(0)) \]

\[ = f(\sqrt{2}/2, \sqrt{2}/2, \pi/4) - f(1, 0, 1) \]

\[ = \frac{\pi}{8} - 0. \]
P16. Compute the flux of the vector field \( X \) across the surface (2-surface) \( M \).

a) \( X = (x, y, z^4) \) where \( M \) is the part of the cone \( z = \sqrt{x^2 + y^2} \) beneath the plane \( z = 1 \).

b) \( X = (xy, yz, xz) \), where \( M \) is the part of the paraboloid \( z = 4 - x^2 - y^2 \) that lies above the square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \).

c) \( X = (x, y, z) \), where \( M \) is the sphere of radius 4 centered at the origin.

Solution 16.

Solution 16a) Answer: \(-\frac{\pi}{3}\)

Parametrizing the cone in cylindrical coordinates by

\[ F(r, \theta) = (r \cos \theta, r \sin \theta, r), \]

with \( 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \), we have tangent vectors

\[ \frac{\partial F}{\partial r} = (\cos \theta, \sin \theta, 1) \]

and

\[ \frac{\partial F}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0). \]

The corresponding normal vector is

\[ \frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} = (-r \cos \theta, -r \sin \theta, r). \]

Since \( X(F) = (r \cos \theta, r \sin \theta, r^4) \), we have

\[ \int_0^{2\pi} \int_0^1 \left( X(F), \frac{\partial F}{\partial r} \times \frac{\partial F}{\partial \theta} \right) dr \ d\theta = \int_0^{2\pi} \int_0^1 (r^5 - r^2) dr \ d\theta = \int_0^{2\pi} -\frac{1}{6} d\theta = \frac{\pi}{3}. \]

Note that the \( z \)-component of the normal vector is positive, hence we have calculated the flux with respect to the \textit{interior} normal (why?).

Now, we could have obtained the same result using the divergence theorem. The divergence of \( X \) is

\[ \text{div} \ X = 2 + 4z^3. \]

The 3-surface \( B \), parametrized by

\[ \psi(r, \theta, z) = (r \cos \theta, r \sin \theta, z), \]
with $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, $r \leq z \leq 1$, has boundary consisting of the cone $M$ and a disc of radius 1 at $z = 1$. The volume form of $B$ is

$$dV_B = r \, dr \wedge d\theta \wedge dz,$$

hence we integrate

$$\int_0^{2\pi} \int_0^1 \int_r^1 (2 + 4z^3) r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (2rz + rz^4) \bigg|_r^1 \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (3r - 2r^2 - r^5) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left( \frac{3}{2} r^2 - \frac{2}{3} r^3 - \frac{1}{6} r^6 \right) \bigg|_0^1 \, d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} \, d\theta$$

$$= \frac{4\pi}{3}.$$

Keep in mind that the divergence theorem assumes a positive (or exterior) orientation. Thus, the total flux across both portions of the boundary, with respect to the exterior normal, is $\frac{4\pi}{3}$.

The top disc corresponds to $z = 1$, $x^2 + y^2 \leq 1$, which can be parametrized by

$$G(r, \theta) = (r \cos \theta, r \sin \theta, 1).$$

The tangent vectors are

$$\frac{\partial G}{\partial r} = (\cos \theta, \sin \theta, 0)$$

and

$$\frac{\partial G}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0).$$

The corresponding normal vector is

$$\frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} = (0, 0, r).$$

Since $X(G) = (r \cos \theta, r \sin \theta, 1)$, we have

$$\int_0^{2\pi} \int_0^1 \left< X(G), \frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} \right> \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta$$

$$= \pi.$$
Hence, the flux across $M$ is $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$, with respect to the exterior normal. This is equivalent to a flux of $-\frac{\pi}{3}$ across $M$ with respect to the interior normal.

**Solution 16b)** *Answer: $\frac{713}{180}$*

**Solution 16c)** *Answer: $256\pi$*