The (Strong) Dynamical Subadditivity for Quantum Channels

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Contents

1 Preliminaries and Notations

- Quantum system: $\mathcal{H}(\dim\mathcal{H} = N < +\infty), \mathbf{L}(\mathcal{H}), \mathbf{L}(\mathcal{H})^{+}$
- Quantum state: $\rho, \mathsf{D}(\mathcal{H})$
- Linear super-operator: Φ, $\mathsf{T}(\mathcal{H}), \mathsf{T}_{\text{cp}}(\mathcal{H}), \mathsf{T}_{\text{cptp}}(\mathcal{H}), \mathsf{T}_{\text{bcptp}}(\mathcal{H})$
- vec mapping: $\mathbf{vec} : \mathbf{L}(\mathcal{H}) \longrightarrow \mathcal{H} \otimes \mathcal{H}$

$$\mathbf{vec}(M) := \sum_{\mu,\nu} m_{\mu\nu} \operatorname{vec}(|\mu\rangle\langle\nu|) = \sum_{\mu,\nu} m_{\mu\nu}|\mu\nu\rangle,$$

  where $M := \sum_{\mu,\nu} m_{\mu\nu}|\mu\rangle\langle\nu|$ is the matrix representation of operator $M \in \mathbf{L}(\mathcal{H})$ w.r.t. a chosen orthonormal basis for $\mathcal{H}$.
- Choi-Jiamiołkowski isomorphism: $J : \mathsf{T}(\mathcal{H}) \longrightarrow \mathbf{L}(\mathcal{H} \otimes \mathcal{H})$

$$J : \Phi \mapsto J(\Phi) := (\Phi \otimes \mathbb{1}_{\mathbf{L}(\mathcal{H})})(\operatorname{vec}(\mathbb{1}_{\mathcal{H}}) \operatorname{vec}(\mathbb{1}_{\mathcal{H}})^{\dagger}), \rho(\Phi) := \frac{1}{N} J(\Phi)$$

- Entropy: $\mathsf{S}(\rho) := -\operatorname{Tr}(\rho \log \rho)$
- Map entropy: $\mathsf{S}^{\text{map}}(\Phi) := -\operatorname{Tr}(\rho(\Phi) \log \rho(\Phi))$

2 Introduction: Two well-known entropic inequalities

❑ **Subadditivity of entropy of bipartite quantum state ρ^{AB}:**

$$\mathrm{S}(\rho^{AB}) \leqslant \mathrm{S}(\rho^{A}) + \mathrm{S}(\rho^{B}).$$

- $\mathrm{S}(\rho^{A}) + \mathrm{S}(\rho^{B}) - \mathrm{S}(\rho^{AB}) \geqslant \frac{1}{2\ln 2}\left(\left\|\rho^{AB} - \rho^{A}\otimes\rho^{B}\right\|_1\right)^2$ (Pinsker's inequality).
- $\mathrm{S}(\rho^{AB}) = \mathrm{S}(\rho^{A}) + \mathrm{S}(\rho^{B}) \Longleftrightarrow \rho^{AB} = \rho^{A}\otimes\rho^{B}$.

❑ **Strong subadditivity of entropy of tripartite quantum state ρ^{ABC}(Lieb-Ruskai):**

$$\mathrm{S}(\rho^{ABC}) + \mathrm{S}(\rho^{B}) \leqslant \mathrm{S}(\rho^{AB}) + \mathrm{S}(\rho^{BC}).$$

- $\mathrm{S}(\rho^{AB}) + \mathrm{S}(\rho^{BC}) = \mathrm{S}(\rho^{ABC}) + \mathrm{S}(\rho^{B})$ iff the following conditions hold:

(i) $\mathcal{H}^B = \bigoplus_k \mathcal{H}^L_k \otimes \mathcal{H}^R_k$,

(ii) $\rho^{ABC} = \bigoplus_k p_k \rho^{AL}_k \otimes \rho^{RC}_k, \rho^{AL}_k \in \mathrm{D}(\mathcal{H}^A \otimes \mathcal{H}^L_k), \rho^{RC}_k \in \mathrm{D}(\mathcal{H}^R_k \otimes \mathcal{H}^C)$ for each index k and $\{p_k\}$ is a probability distribution (Hayden).

3 Motivation

- **Theorem(Roga08):** $\Phi, \Lambda, \Psi \in \mathsf{T}_{\text{cptp}}(\mathcal{H})$.
  - *Dynamical Subadditivity*:

$$\mathsf{S}^{\text{map}}(\Phi \circ \Psi) \leqslant \mathsf{S}^{\text{map}}(\Phi) + \mathsf{S}^{\text{map}}(\Psi)$$

for $\Phi \in \mathsf{T}_{\text{bcptp}}(\mathcal{H})$;

  - *Strong Dynamical Subadditivity*:

$$\mathsf{S}^{\text{map}}(\Phi \circ \Lambda \circ \Psi) + \mathsf{S}^{\text{map}}(\Lambda) \leqslant \mathsf{S}^{\text{map}}(\Phi \circ \Lambda) + \mathsf{S}^{\text{map}}(\Lambda \circ \Psi)$$

for $\Phi, \Lambda, \Psi \in \mathsf{T}_{\text{bcptp}}(\mathcal{H})$.

4 Main Results

❑ **Theorem I:** Let $\Phi, \Psi \in \mathsf{T}_{\mathrm{bcptp}}(\mathcal{H})$. $\Phi(\rho) = \sum_{m=1}^{N^2} Ad_{S_m}$ and $\Psi = \sum_{\mu=1}^{N^2} Ad_{T_\mu}$ be their canonical representations, respectively. Then

$$\begin{aligned} & \mathsf{S}^{\mathrm{map}}(\Phi \circ \Psi) = \mathsf{S}^{\mathrm{map}}(\Phi) + \mathsf{S}^{\mathrm{map}}(\Psi) \\ \Longleftrightarrow\ & \mathrm{Tr}(S_m T_\mu (S_n T_\nu)^\dagger) = \frac{1}{N} \mathrm{Tr}(S_m S_n^\dagger)\,\mathrm{Tr}(T_\mu T_\nu^\dagger) \\ \Longleftrightarrow\ & \langle S_n T_\nu, S_m T_\mu \rangle = \frac{1}{N} \langle S_n, S_m \rangle \langle T_\nu, T_\mu \rangle, \forall m, n, \mu, \nu = 1, \ldots, N^2. \end{aligned}$$

❑ **Theorem II:** $\Phi, \Lambda, \Psi \in \mathsf{T}_{\rm bcptp}(\mathcal{H})$. And following conditions hold:

(i) $\mathcal{H} = \bigoplus_{k=1}^{K} \mathcal{H}_k^L \otimes \mathcal{H}_k^R$, $\dim \mathcal{H}_k^L = d_k^L$, $\dim \mathcal{H}_k^R = d_k^R$, $\sum_{k=1}^{K} d_k^L d_k^R = N$;

(ii) $\Phi = \bigoplus_{k=1}^{K} \Phi_k^L \otimes Ad_{U_k^R}$, $\Lambda = \bigoplus_{k=1}^{K} \Lambda_k^L \otimes \Lambda_k^R$, $\Psi = \bigoplus_{k=1}^{K} Ad_{V_k^L} \otimes \Psi_k^R$.
That is, $\Phi|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = \Phi_k^L \otimes Ad_{U_k^R}$, $\Psi|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = Ad_{V_k^L} \otimes \Psi_k^R$, $\Lambda|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = \Lambda_k^L \otimes \Lambda_k^R$, where $\Phi_k^L, \Lambda_k^L \in \mathsf{T}_{\rm bcptp}(\mathcal{H}_k^L)$, $V_k^L \in \mathbf{L}(\mathcal{H}_k^L)$ are unitary operators, $U_k^R \in \mathbf{L}(\mathcal{H}_k^R)$ are unitary operators and $\Psi_k^R, \Lambda_k^R \in \mathsf{T}_{\rm bcptp}(\mathcal{H}_k^R)$.

Then we have the following strong dynamical additivity:

$$\mathsf{S}^{\rm map}(\Phi \circ \Lambda) + \mathsf{S}^{\rm map}(\Lambda \circ \Psi) = \mathsf{S}^{\rm map}(\Lambda) + \mathsf{S}^{\rm map}(\Phi \circ \Lambda \circ \Psi).$$

5 Proofs

Definition 1. $S, T \in \mathbf{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)^+$ are said to be *bi-orthogonal* if

$$\mathrm{Tr}_1(S)\,\mathrm{Tr}_2(S) = \mathrm{Tr}_1(T)\,\mathrm{Tr}_2(T) = 0.$$

Definition 2(Herbut). $\rho^{AB} \in \mathsf{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$. The following state decomposition $\rho^{AB} = \sum_k p_k \rho_k^{AB}, \forall k : \rho_k^{AB} \in \mathsf{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$ and $\{p_k\}$ is a probability distribution, is called *bi-orthogonal* if, in terms of the reductions of ρ_k^{AB}, $\rho_k^X \rho_{k'}^X = 0(X = A, B; \forall k \neq k')$.

Definition 3. $(\mathcal{H}^A, \{|m\rangle\}), (\mathcal{H}^B, \{|\mu\rangle\})$. vec mapping may be defined over a bipartite space $\mathcal{H}^A \otimes \mathcal{H}^B$, that describes a change of bases from the standard basis of $\mathbf{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$ to the standard basis of $\mathcal{H}^A \otimes \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^B$:

$$\mathbf{vec}(|m\rangle\langle n| \otimes |\mu\rangle\langle \nu|) = |mn\rangle \otimes |\mu\nu\rangle = |mn\mu\nu\rangle.$$

- The following properties of the vec map are easily verified:
  - The **vec** map is a linear bijection. It is also an isometry, in the sense that

$$\langle X, Y\rangle = \langle \mathrm{vec}(X), \mathrm{vec}(Y)\rangle$$

for all $X, Y \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$.

  - For every choice of operators $A \in \mathbf{L}(\mathcal{H}_1, \mathcal{K}_1), B \in \mathbf{L}(\mathcal{H}_2, \mathcal{K}_2)$, and $X \in \mathbf{L}(\mathcal{H}_2, \mathcal{H}_1)$, it holds that

$$(A \otimes B)\,\mathrm{vec}(X) = \mathrm{vec}(AXB^{\mathsf{T}}).$$

  - For every choice of operators $A, B \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$, the following equations hold:

$$\begin{aligned} \mathrm{Tr}_1(\mathrm{vec}(A)\,\mathrm{vec}(B)^{\dagger}) &= AB^{\dagger}, \\ \mathrm{Tr}_2(\mathrm{vec}(A)\,\mathrm{vec}(B)^{\dagger}) &= (B^{\dagger}A)^{\mathsf{T}}. \end{aligned}$$

  - If $X \in \mathbf{L}(\mathcal{H}_A), Z \in \mathbf{L}(\mathcal{H}_B)$, then $\mathrm{vec}(X \otimes Z) = \mathrm{vec}(X) \otimes \mathrm{vec}(Z)$.

Theorem(Choi75): $\Phi \in \mathsf{T}_{\mathrm{cp}}(\mathcal{H})$ can be represented in the following form:

$$\Phi = \sum_{k=1}^{K} \mathbf{Ad}_{M_k}, M_k \in \mathbf{L}(\mathcal{H}), K \leqslant (\dim \mathcal{H})^2,$$

where $\mathbf{Ad}_{M_k}(X) := M_k X M_k^{\dagger}$.

When M_k is a collection of orthogonal operators, the above expression is referred to a *canonical representation* for Φ.

Entropy-Preserving Extensions of Quantum States and the Dynamical Additivity

- $\Lambda \in \mathsf{T}_{\text{cptp}}(\mathcal{H})$. If Λ has two Kraus representations $\Lambda = \sum_{p=1}^{d_1} Ad_{S_p} = \sum_{q=1}^{d_2} Ad_{T_q}$, $\rho \in \mathsf{D}(\mathcal{H})$, take two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that dim $\mathcal{H}_1 = d_1$, dim $\mathcal{H}_2 = d_2$, $\{|m\rangle\}$ and $\{|\mu\rangle\}$ are the base of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Define

$$\gamma_1(\Lambda) = \sum_{m,n=1}^{d_1} \mathrm{Tr}(S_m \rho S_n^\dagger)|m\rangle\langle n| \text{ and } \gamma_2(\Lambda) = \sum_{\mu,\nu=1}^{d_2} \mathrm{Tr}(T_\mu \rho T_\nu^\dagger)|\mu\rangle\langle \nu|.$$

Then $\gamma_k \in \mathsf{D}(\mathcal{H}_k)(k = 1, 2)$, and $\mathsf{S}(\gamma_1(\Lambda)) = \mathsf{S}(\gamma_2(\Lambda))$.

If we denote $\mathsf{S}(\rho; \Lambda)$ by $\mathsf{S}(\gamma_1(\Lambda))$, then $\mathsf{S}(\rho; \Lambda)$ is well-defined. Moreover, it is easy to see that if $\rho = \frac{1}{N}\mathbb{1}$, then $\mathsf{S}(\rho; \Lambda) = \mathsf{S}^{\text{map}}(\Lambda)$.

• $\Phi, \Psi \in \mathsf{T}_{\text{bcptp}}(\mathcal{H})$. $\Phi = \sum_{m=1}^{N^2} Ad_{S_m}$ and $\Psi = \sum_{\mu=1}^{N^2} Ad_{T_\mu}$ are their canonical representations, respectively. Taking a N^2 dimensional complex Hilbert space $\mathcal{H}_0$, for each $\rho \in \mathsf{D}(\mathcal{H})$, we define

$$\gamma(\Phi \circ \Psi) = \sum_{m,n,\mu,\nu=1}^{N^2} \mathrm{Tr}(S_m T_\mu \rho (S_n T_\nu)^\dagger) |m\mu\rangle\langle n\nu|,$$

then $\gamma(\Phi \circ \Psi)$ is a state on $\mathcal{H}_0 \otimes \mathcal{H}_0$, and when $\rho = \frac{1}{N}\mathbb{1}$, $\mathsf{S}(\gamma(\Phi \circ \Psi)) = \mathsf{S}^{\text{map}}(\Phi \circ \Psi)$, that is, $\mathsf{S}(\rho, \Phi \circ \Psi) = \mathsf{S}^{\text{map}}(\Phi \circ \Psi)$.

$$\begin{aligned} \gamma(\Psi) &= \sum_{\mu,\nu=1}^{N^2} \mathrm{Tr}(T_\mu \rho T_\nu^\dagger) |\mu\rangle\langle \nu| = \mathrm{Tr}_1(\gamma(\Phi \circ \Psi)), \\ \gamma(\Phi) &= \sum_{m,n=1}^{N^2} \mathrm{Tr}(S_m \rho S_n^\dagger)) |m\rangle\langle n| = \mathrm{Tr}_2(\gamma(\Phi \circ \Psi)). \end{aligned}$$

$$\mathsf{S}^{\text{map}}(\Phi \circ \Psi) = \mathsf{S}^{\text{map}}(\Phi) + \mathsf{S}^{\text{map}}(\Psi) \Leftrightarrow \gamma(\Phi \circ \Psi) = \gamma(\Phi) \otimes \gamma(\Psi)$$

Bi-orthogonal Decomposition and Strong Dynamical Additivity

- $\Phi, \Psi \in \mathsf{T}_{\mathrm{cp}}(\mathcal{H})$. If $J(\Phi)$ and $J(\Psi)$ are bi-orthogonal, then Φ and Ψ are said to be *bi-orthogonal*.

- **Definition.** $\Phi \in \mathsf{T}_{\mathrm{cp}}(\mathcal{H})$ has a *bi-orthogonal decomposition* if $J(\Phi)$ has a bi-orthogonal decomposition: $J(\Phi) = \sum_k D_k$, where $\{D_k\}$ is a family of pairwise bi-orthogonal positive semi-definite operators.

- If $\Phi = \sum_\mu Ad_{M_\mu}$, $\Psi = \sum_\nu Ad_{N_\nu}$, then Φ and Ψ are bi-orthogonal if and only if $M_\mu^\dagger N_\nu = 0$ and $M_\mu N_\nu^\dagger = 0$ for all μ and ν, if and only if $\Phi \circ \Psi^\dagger = 0$ and $\Phi^\dagger \circ \Psi = 0$, if and only if $\Psi \circ \Phi^\dagger = 0$ and $\Psi^\dagger \circ \Phi = 0$.

- $\Phi \in \mathsf{T}_{\mathrm{cp}}(\mathcal{H})$ has a bi-orthogonal decomposition if and only if $\Phi = \sum_k \Phi_k$, where $\{\Phi_k\}$ is a collection of super-operators from $\mathsf{T}_{\mathrm{cp}}(\mathcal{H})$ and ${\Phi_m}^\dagger \circ \Phi_n = 0$ and $\Phi_m \circ {\Phi_n}^\dagger = 0$ for all $m \neq n$.

6 Concluding Remarks

- The References can be referred to [Zhang and Wu, arXiv:1104.2193(to appear in J. Phys. A)].
- Q: What is a necessary condition for Theorem II?
- Comparison with the *Squashed Entanglement* proposed recently by Christandl:

$$E_{sq}(\rho^{AB}) = \inf_E \{\frac{1}{2} I(A;B|E) : \rho^{ABE} \text{ extension of } \rho^{AB}\},$$

  where

$$I(A;B|E) = \mathrm{S}(\rho^{AE}) + \mathrm{S}(\rho^{BE}) - \mathrm{S}(\rho^{ABE}) - \mathrm{S}(\rho^{E})$$

  is the quantum conditional mutual information of ρ^{ABE}, which measures the correlations of two quantum systems relative to a third one.
- Consider the following quantity:

$$I(\Phi;\Psi|\Lambda) = \mathrm{S}(\Phi\circ\Lambda) + \mathrm{S}(\Lambda\circ\Psi) - \mathrm{S}(\Phi\circ\Lambda\circ\Psi) - \mathrm{S}(\Lambda)$$

Thank you!