Fidelity of states in infinite dimensional quantum systems

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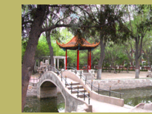

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1 Introduction

- In quantum mechanics, a quantum system is associated with a separable complex Hilbert space H, i.e., the state space.
- $\mathcal{B}(H)$ denotes the algebra of all bounded linear operators on H.
- $\mathcal{T}(H) \subseteq \mathcal{B}(H)$ denotes the algebra of the trace-class of all operators T with $\|T\|_{\mathrm{Tr}} = \mathrm{Tr}((T^\dagger T)^{\frac{1}{2}}) < \infty$.
- A quantum state is described as a density operator $\rho \in \mathcal{T}(H) \subseteq \mathcal{B}(H)$ which is positive and has trace 1.
- Denote by $\mathcal{S}(H)$ the set of all states acting on H.

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- The fidelity of states ρ and σ in $\mathcal{S}(H)$ is defined to be

$$F(\rho,\sigma) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}. \tag{1.1}$$

  Fidelity is a very useful measure of closeness between two states and has several nice properties.

- Uhlmann and co-workers developed Eq.(1.1) by the transition probability in the more general context of the representation theory of C*-algebras. The result in

  [1] Uhlmann A., The 'transition probability' in the state space of a *-algebra, Rep. Math. Phys., 9, 273(1976).

  implies that, if $\dim H < \infty$, then the equality

$$F(\rho,\sigma) = \max |\langle\psi|\phi\rangle|, \tag{1.2}$$

  holds, where the maximum is over all purifications $|\psi\rangle$ of ρ and $|\phi\rangle$ of σ into a larger system of $H \otimes H$.

- Recall that a unit vector $|\psi\rangle \in H \otimes K$ is said to be a purification of a state ρ on H if $\rho = \mathrm{Tr}_K(|\psi\rangle\langle\psi|) = (I_H \otimes \mathrm{Tr})(|\psi\rangle\langle\psi|)$.

- This result is then referred as the Uhlmann's theorem.
- Eq.(1.2) does not provide a calculation tool for evaluating the fidelity, as does Eq.(1.1). However, in many instances, the properties of the fidelity are more easily deduced using Eq.(1.2) than Eq.(1.1). For example, Eq.(1.2) makes it clear that $0 \leq F(\rho, \sigma) = F(\sigma, \rho) \leq 1$; $F(\rho, \sigma) = 1$ if and only if $\rho = \sigma$.
- In

  [2] Jozsa R., Fidelity for mixed quantum states, Journal of Modern Optics, 41(12), 2315(1994).

  Jozsa presented an elementary proof of the Uhlmann's theorem without involving the representation theory of C*-algebras.

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- In this paper we will consider the fidelity of states in infinite dimensional systems, give an elementary proof of the infinite dimensional version of Uhlmann's theorem, and then, apply it to generalize several properties of the fidelity from finite dimensional case to infinite dimensional case.

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2 Infinite dimensional version of the Uhlmann's theorem

- Recall that an operator $V \in \mathcal{B}(H)$ is called an isometry if $V^\dagger V = I$; is called a co-isometry if $VV^\dagger = I$.
- If $\dim H = \infty$ and $T \in \mathcal{B}(H)$, then, by the polar decomposition, there exists an isometry or a co-isometry V such that $T = V|T|$, where $|T| = (T^\dagger T)^{1/2}$. Generally speaking, V may not be unitary.
- In fact, there exists a unitary operator U such that $T = U|T| \Leftrightarrow \dim\ker T = \dim\ker T^\dagger$.
- However, the following lemma says that it is the case if T is a product of two positive operators.

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Lemma 2.1. *Let H be a Hilbert space and $A, B \in \mathcal{B}(H)$. If $A \geq 0$ and $B \geq 0$, then there exists a unitary operator $V \in \mathcal{B}(H)$ such that $AB = V|AB|$.*

- This lemma is crucial for our purpose.

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- If $\dim H < \infty$, then, for any $T \in \mathcal{B}(H)$, we have

$$\|T\|_{\mathrm{Tr}} = \mathrm{Tr}(|T|) = \max_{U}\{\mathrm{Tr}(AU)\},$$

  where the maximum is over all unitary operators.

- This result is not valid even for trace-class operators if $\dim H = \infty$.

- The next lemma says that the above result is true if the operator is a product of two positive operators.

Lemma 2.2. *Let H be a complex Hilbert space and $A, B \in \mathcal{B}(H)$. If A, B are positive and $AB \in \mathcal{T}(H)$, then*

$$\|AB\|_{\mathrm{Tr}} = \mathrm{Tr}(|AB|) = \max\{\mathrm{Tr}(ABU) : U \in \mathcal{U}(H)\},$$

where $\mathcal{U}(H)$ is the unitary group of all unitary operators in $\mathcal{B}(H)$.

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- By Lemmas 2.1-2.2, we can prove the following infinite dimensional version of the Uhlmann's theorem.

Theorem 2.3. *Let H and K be separable infinite dimensional complex Hilbert spaces. For any states ρ and σ on H, we have*

$$F(\rho, \sigma) = \max\{|\langle\psi|\phi\rangle| : |\psi\rangle \in \mathcal{P}_\rho,\ |\phi\rangle \in \mathcal{P}_\sigma\},$$

where $\mathcal{P}_\rho = \{|\psi\rangle \in H \otimes K : |\psi\rangle$ is a purification of $\rho\}$.

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- By checking the proof of Theorem 2.3, it is easily seen that the following holds.

Corollary 2.4. *Let H and K be separable infinite dimensional complex Hilbert spaces. For any states ρ and σ on H, we have*

$$\begin{aligned} F(\rho,\sigma) &= \max\{|\langle\psi_0|\phi\rangle| : |\phi\rangle \in \mathcal{P}_\sigma\} \\ &= \max\{|\langle\psi|\phi_0\rangle| : |\psi\rangle \in \mathcal{P}_\rho\}, \end{aligned}$$

where $|\psi_0\rangle$ is any fixed purification of ρ of the form $|\psi_0\rangle = \sum_{i=1}^{\infty}\sqrt{p_i}|i_H\rangle|i_K\rangle$ with $\{|i_K\rangle\}$ an orthonormal basis of K, and $|\phi_0\rangle$ is any fixed purification of σ of the form $|\phi_0\rangle = \sum_{i=1}^{\infty}\sqrt{q_i}|i'_H\rangle|i'_K\rangle$ with $\{|i'_K\rangle\}$ an orthonormal basis of K.

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- The fidelity is not a distance because it does not meet the triangular inequality. However, like to the finite dimensional case, by use of Theorem 2.3 and Corollary 2.4, one can show that the arc-cosine of fidelity is a distance.

  Corollary 2.5. $A(\rho, \sigma) =: \arccos F(\rho, \sigma)$ *is a distance on* $\mathcal{S}(H)$.

- Several remarkable properties of fidelity in finite dimensional case are still valid for infinite dimensional case. For instance,

  Monotonicity of the fidelity For any quantum channel $\mathcal{E}$, we have

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma).$$

  Recall that a quantum channel is a completely positive and trace preserving linear map from $\mathcal{T}(H)$ into $\mathcal{T}(K)$.

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3 Connection to the classical fidelity and trace distance

- If $\dim H < \infty$, the quantum fidelity is related to the classical fidelity by considering the probability distributions induced by a measurement. In fact,

$$F(\rho, \sigma) = \min_{\{E_m\}} F(p_m, q_m), \tag{3.1}$$

where the minimum is over all POVMs (positive operator-valued measure) $\{E_m\}$, and $p_m = \mathrm{Tr}(\rho E_m)$, $q_m = \mathrm{Tr}(\sigma E_m)$ are the probability distributions for ρ and σ corresponding to the POVM $\{E_m\}$.

- It is natural to ask whether or not Eq.(3.1) is true if $\dim H = \infty$? The following result is our answer.

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Theorem 3.1. *Let H be a separable infinite dimensional complex Hilbert space. Then, for any states $\rho, \sigma \in \mathcal{S}(H)$, we have*

$$F(\rho, \sigma) = \inf_{\{E_m\}} F(p_m, q_m), \tag{3.2}$$

where the infimum is over all POVMs $\{E_m\}_{m=1}^{\infty}$, and $p_m = \operatorname{Tr}(\rho E_m)$, $q_m = \operatorname{Tr}(\sigma E_m)$ are the probability distributions for ρ and σ corresponding to the POVM $\{E_m\}$. Moreover, the infimum attains the minimum if and only if the operator $M = \rho^{[-1/2]}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\rho^{[-1/2]}$ (may unbounded) is diagonal.

- Here, for a positive operator $A \in \mathcal{B}(H)$, with respect to the space decomposition $H = (\ker A)^{\perp} \oplus \ker A$, we denote $A^{[-1]}$ for the may unbounded densely defined positive operator defined by $A^{[-1]} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ with domain $\mathcal{D}(A^{[-1]}) = \operatorname{ran}(A) \oplus \ker A$.

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- In order to prove Theorem 3.1, we give a lemma.

Lemma 3.2. *Let H be an infinite dimensional complex Hilbert space. Assume that $A \in \mathcal{T}(H)$ and $\{T_n\}_{n=0}^{\infty} \subset \mathcal{B}(H)$. If SOT-$\lim_{n\to\infty} T_n = T_0$, then $\lim_{n\to\infty} \mathrm{Tr}(T_n A) = \mathrm{Tr}(T_0 A)$. Here SOT means the strong operator topology.*

Remark 3.3. There do exist some ρ and σ such that there is no POVM $\{E_m\}$ satisfying $F(\rho, \sigma) = \sum_m \sqrt{\mathrm{Tr}(\rho E_m)\mathrm{Tr}(\sigma E_m)}$. For example, let $H = L_2([0,1])$ and M_t the operator defined by $(M_t f)(t) = t f(t)$ for any $f \in H$. Then M_t is positive and is not diagonal because $\sigma(M_t) = [0,1]$ and the point spectrum $\sigma_p(M_t) = \emptyset$. Let $\rho \in \mathcal{S}(H)$ be injective as an operator. Then $d = \mathrm{Tr}(M_t^2 \rho) \neq 0$. Let $M = d^{-1} M_t$ and $\sigma = M \rho M$. As $\mathrm{Tr}(M^2 \rho) = 1$, σ is a state. Now it is clear that $M = \rho^{-1/2}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\rho^{-1/2}$, which is not diagonal. Thus by Theorem 3.1, the infimum in Eq.(3.2) does not attain the minimum.

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- For two states ρ and σ, recall that the trace distance of them is defined by $D(\rho,\sigma) = \frac{1}{2}\|\rho - \sigma\|_{\mathrm{Tr}}$. By use of Uhlmann's theorem and Eq.(3.1), it holds for finite dimensional case that

$$1 - F(\rho,\sigma) \leq D(\rho,\sigma) \leq \sqrt{1 - F(\rho,\sigma)^2}. \qquad (3.3)$$

This reveals that the trace distance and the fidelity are qualitatively equivalent measures of closeness for quantum states. Now Theorem 3.1 allows us to establish the same relationship between fidelity measure and trace distance measure for states of infinite dimensional systems.

Theorem 3.4. *Let H be an infinite dimensional separable complex Hilbert space. Then for any states $\rho, \sigma \in \mathcal{S}(H)$, the inequalities in Eq.(3.3) hold.*

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4 Fidelities connected to channels

- For finite dimensional case, ensemble average fidelity and entanglement fidelity are two kinds of important fidelities connected to a quantum channel. In this section we give the definitions of ensemble average fidelity and entanglement fidelity connected to a quantum channel for an infinite dimensional system, and discuss their relationship.

- Let H be an infinite dimensional separable complex Hilbert space. Like the finite dimensional case, for a quantum channel $\mathcal{E}$ and a given ensemble $\{p_j, \rho_j\}_{j=1}^{\infty}$, one can define ensemble average fidelity by

$$\overline{F} = \sum_j p_j F(\rho_j, \mathcal{E}(\rho_j))^2. \tag{4.1}$$

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- Similarly, for a state ρ, one can define the entanglement fidelity by

$$\begin{aligned} F(\rho, \mathcal{E}) &= F(|\psi\rangle, (\mathcal{E} \otimes I)(|\psi\rangle\langle\psi|))^2 \\ &= \langle\psi|(\mathcal{E} \otimes I)(|\psi\rangle\langle\psi|)|\psi\rangle, \end{aligned} \tag{4.2}$$

where $|\psi\rangle \in H \otimes H$ is a purification of ρ.

- Note that the definition $F(\rho, \mathcal{E})$ does not depend on the choices of purifications. In fact, since there exists a sequence of operators $\{E_i\} \subseteq \mathcal{B}(H)$ with $\sum_i E_i^\dagger E_i = I$ such that

$$\mathcal{E}(\sigma) = \sum_i E_i \sigma E_i^\dagger \quad \text{for all} \quad \sigma \in \mathcal{S}(H),$$

we can prove that

$$F(\rho, \mathcal{E}) = \sum_i |\mathrm{Tr}(E_i \rho)|^2, \tag{4.3}$$

which is dependent only to ρ and $\mathcal{E}$.

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- In the sequel we will give some properties of entanglement fidelity for infinite dimensional systems.

- Firstly note that, by monotonicity of the fidelity, it is easily checked that

$$F(\rho, \mathcal{E}) \leq [F(\rho, \mathcal{E}(\rho))]^2.$$

Proposition 4.1. *Let H be an infinite dimensional separable complex Hilbert space. Assume that $\mathcal{E} : \mathcal{T}(H) \to \mathcal{T}(H)$ is a quantum channel and $\rho \in \mathcal{S}(H)$. Then the entanglement fidelity $F(\rho, \mathcal{E})$ is a convex function of ρ.*

Proposition 4.2. *Let H be an infinite dimensional separable complex Hilbert space. Assume that $\mathcal{E} : \mathcal{T}(H) \to \mathcal{T}(H)$ is a quantum channel. Then for any given ensemble $\{p_j, \rho_j\}$, we have $F(\sum_j p_j \rho_j, \mathcal{E}) \leq \overline{F}$.*

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Thanks!

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