THE DYNAMICAL ADDITIVITY AND THE STRONG DYNAMICAL ADDITIVITY OF QUANTUM OPERATIONS

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Abstract. In the paper, firstly, by using the methods of entropy-preserving extensions of quantum states, the dynamical additivity of bi-stochastic quantum operations is characterized. Next, we show that if quantum operations are local operations and have some orthogonality, then the strong dynamical additivity is true, too.

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1. Introduction

In this paper, we always assume that $\mathcal{H}$ is an N-dimensional complex Hilbert space. Let $\mathbf{L}(\mathcal{H})$ be the set of all linear operators from $\mathcal{H}$ to $\mathcal{H}$. A state ρ of some quantum system, described by $\mathcal{H}$, is a positive semi-definite operator of trace one, in particular, for each unit vector $|\psi\rangle \in \mathcal{H}$, the operator $\rho = |\psi\rangle\langle\psi|$ is said to be a *pure state*. The set of all states on $\mathcal{H}$ is denoted by $\mathsf{D}(\mathcal{H})$. If $X, Y \in \mathbf{L}(\mathcal{H})$, then $\langle X, Y\rangle = \mathrm{Tr}(X^\dagger Y)$ defines an inner product on $\mathbf{L}(\mathcal{H})$, which is called the *Hilbert-Schmidt inner product*. The following fact is often used:

If $X, Y \in \mathbf{L}(\mathcal{H})$ are two positive semi-definite operators, it follows from $\langle X, Y\rangle = \mathrm{Tr}(X^\dagger Y) = \mathrm{Tr}(XY) = \mathrm{Tr}(X^{\frac{1}{2}} Y X^{\frac{1}{2}})$ that $\langle X, Y\rangle = 0$ if and only if $XY = 0$, that is, X and Y are orthogonal if and only if $XY = 0$.

Let $S, T \in \mathbf{L}(\mathcal{H} \otimes \mathcal{H})$ be two positive semi-definite operators. Denote $S_1 = \mathrm{Tr}_2(S)$, $T_1 = \mathrm{Tr}_2(T)$, $S_2 = \mathrm{Tr}_1(S)$ and $T_2 = \mathrm{Tr}_1(T)$. Then $S_1, T_1, S_2, T_2 \in \mathbf{L}(\mathcal{H})$ are all positive semi-definite operators. If $S_1T_1 = S_2T_2 = 0$, then S and T is said to be *bi-orthogonal* (see [7]).

Let $\{|i\rangle\}$ be the standard basis of $\mathcal{H}$. For each $P = \sum_{i,j} p_{ij}|i\rangle\langle j| \in \mathbf{L}(\mathcal{H})$, we denote $\mathrm{vec}(P) = \sum_{ij} p_{ij}|ij\rangle$, then vec defined a linear map from $\mathbf{L}(\mathcal{H})$ to $\mathcal{H} \otimes \mathcal{H}$. Moreover, if $\mathcal{H}_A$ and $\mathcal{H}_B$ are two Hilbert spaces, $\{|m\rangle\}$ and $\{|\mu\rangle\}$ are their standard bases, respectively, then we can also define a map vec that describes a change of the standard basis from $\mathbf{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ to $\mathcal{H}_A \otimes \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_B$, that is, $\mathrm{vec}(|m\rangle\langle n| \otimes |\mu\rangle\langle\nu|) = |mn\rangle \otimes |\mu\nu\rangle$. Moreover, if $X \in \mathbf{L}(\mathcal{H}_A)$, $Z \in \mathbf{L}(\mathcal{H}_B)$, then $\mathrm{vec}(X \otimes Z) = \mathrm{vec}(X) \otimes \mathrm{vec}(Z)$ (see [14]).

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Let $\mathsf{T}(\mathcal{H})$ denote the set of all *linear super-operators* from $\mathbf{L}(\mathcal{H})$ to $\mathbf{L}(\mathcal{H})$. For each $\Phi \in \mathsf{T}(\mathcal{H})$, it follows from the Hilbert-Schmidt inner product of $\mathbf{L}(\mathcal{H})$ that there is a linear super-operator $\Phi^\dagger \in \mathsf{T}(\mathcal{H})$ such that $\langle \Phi(X), Y\rangle = \langle X, \Phi^\dagger(Y)\rangle$ for any $X, Y \in \mathbf{L}(\mathcal{H})$. $\Phi^\dagger$ is said to be the *dual super-operator* of Φ.

We say that $\Phi \in \mathsf{T}(\mathcal{H})$ is *completely positive* (CP) if for each $k \in \mathbb{N}$, $\Phi \otimes \mathbb{1}_{M_k(\mathbb{C})} : \mathbf{L}(\mathcal{H}) \otimes M_k(\mathbb{C}) \to \mathbf{L}(\mathcal{H}) \otimes M_k(\mathbb{C})$ is positive, where $M_k(\mathbb{C})$ is the set of all $k \times k$ complex matrices. It follows from the famous theorems of Choi [2] and Kraus [9] that Φ can be represented in the following form: $\Phi = \sum_j Ad_{M_j}$, where $\{M_j\}_{j=1}^n \subseteq \mathbf{L}(\mathcal{H})$, that is, $\Phi(X) = \sum_{j=1}^n M_j X M_j^\dagger$, $X \in \mathbf{L}(\mathcal{H})$. Throughout this paper, † means the adjoint operation of an operator. Moreover, if $\{M_j\}_{j=1}^n$ is pairwise orthogonal, then $\Phi = \sum_j Ad_{M_j}$ is said to be a canonical representation of Φ. In [2, 8], it was proved that each quantum operation has a canonical representation.

The so-called *quantum operation* of $\mathcal{H}$ is just a CP trace non-increasing $\Phi \in \mathsf{T}(\mathcal{H})$, moreover, if Φ is CP and trace-preserving, then it is called *stochastic*; if Φ is stochastic and unit-preserving, then it is called *bi-stochastic*.

The famous *Jamiołkowski isomorphism* $J : \mathsf{T}(\mathcal{H}) \longrightarrow \mathbf{L}(\mathcal{H} \otimes \mathcal{H})$ transforms each $\Phi \in \mathsf{T}(\mathcal{H})$ into an operator $J(\Phi) \in \mathbf{L}(\mathcal{H} \otimes \mathcal{H})$, where $J(\Phi) = \Phi \otimes \mathbb{1}_{\mathbf{L}(\mathcal{H})}(\mathrm{vec}(\mathbb{1}_\mathcal{H})\,\mathrm{vec}(\mathbb{1}_\mathcal{H})^\dagger)$. If $\Phi \in \mathsf{T}(\mathcal{H})$ is CP, then $J(\Phi)$ is a positive semi-definite operator, in particular, if Φ is stochastic, then $\frac{1}{N}J(\Phi)$ is a state on $\mathcal{H} \otimes \mathcal{H}$, we denote the state by $\rho(\Phi)$ (see [1]).

The information encoded in a quantum state $\rho \in \mathsf{D}(\mathcal{H})$ is quantified by its *von Neumann entropy* $\mathsf{S}(\rho) = -\operatorname{Tr}(\rho \log_2 \rho)$. If $\Phi \in \mathsf{T}(\mathcal{H})$ is a stochastic quantum operation, we denote the von Neumann entropy $\mathsf{S}(\rho(\Phi))$ of $\rho(\Phi)$ by $\mathsf{S}(\Phi)$ and call it the *map entropy*, $\mathsf{S}(\Phi)$ describes the decoherence induced by the quantum operation Φ.

Let Φ, Λ and Ψ be three stochastic quantum operations of $\mathcal{H}$. Studying the behavior of map entropy of composition of stochastic quantum operations is an important and interesting problem. In [12], Roga *et. al.* showed that if Φ is bi-stochastic, then ones have the *dynamical subadditivity*:

$$\mathsf{S}(\Phi \circ \Psi) \leqslant \mathsf{S}(\Phi) + \mathsf{S}(\Psi).$$

Moreover, if Φ, Λ and Ψ are all bi-stochastic, then the *strong dynamical subadditivity* holds:

$$\mathsf{S}(\Phi \circ \Lambda \circ \Psi) + \mathsf{S}(\Lambda) \leqslant \mathsf{S}(\Phi \circ \Lambda) + \mathsf{S}(\Lambda \circ \Psi).$$

In [6], the main results described the structure of states that saturate the inequality of strong subadditivity of quantum entropy. Now, we study the saturation problems of the dynamical subadditivity and the strong dynamical subadditivity. Firstly, by using the methods of entropy-preserving extensions of quantum states, a nice characterization of dynamical additivity of bi-stochastic quantum operations is obtained. Next, we show that if Φ, Λ and Ψ are some special local operations [3, 4] and have some orthogonality, then the strong dynamical additivity is true, too.

2. Entropy-preserving Extensions of Quantum States and the dynamical additivity

The technique of quantum state extension without changing entropy is a very important and useful tool. It is employed by Datta to construct an example which shows equivalence of the positivity of quantum discord and strong subadditivity for quantum mechanical systems. Based on this fact, Datta obtained that zero discord states are precisely those states which satisfy the strong additivity for quantum mechanical systems. For the details, it is referred to [5]. In what follows, we will use it to give a characterization of dynamical additivity of map entropy.

For each state ρ on $\mathcal{H}$, we entropy-preserving extend ρ to a state on $\mathcal{H}\otimes\mathcal{H}$, that is, if $\{|i\rangle\}$ is a basis of $\mathcal{H}$ and $\rho = \sum_{i,j=1}^N \rho_{i,j}|i\rangle\langle j|$, then $\widetilde{\rho} = \sum_{i,j=1}^N \rho_{i,j}|ii\rangle\langle jj|$ is a state on $\mathcal{H}\otimes\mathcal{H}$, and $\mathsf{S}(\widetilde{\rho}) = \mathsf{S}(\rho)$.

In fact, by the spectral decomposition theorem, $\rho = \sum_k \lambda_k|x_k\rangle\langle x_k|$, where $\lambda_k \geqslant 0$, $\{|x_k\rangle\}$ is an orthonormal set of $\mathcal{H}$. This implies that $\rho_{i,j} = \langle i|\rho|j\rangle = \sum_k \lambda_k\langle i|x_k\rangle\langle x_k|j\rangle = \sum_k \lambda_k x_k^{(i)}\bar{x}_k^{(j)}$. Note that $\{|x_k\rangle\}$ is an orthonormal set of $\mathcal{H}$, so $\sum_{i=1}^N x_m^{(i)}\bar{x}_n^{(i)} = \delta_{mn}$. Now

$$\begin{aligned}\widetilde{\rho} &= \sum_{i,j=1}^N(\sum_k \lambda_k x_k^{(i)}\bar{x}_k^{(j)})|i\rangle\langle j|\otimes|i\rangle\langle j| = \sum_k \lambda_k(\sum_{i,j=1}^N x_k^{(i)}\bar{x}_k^{(j)}|i\rangle\langle j|\otimes|i\rangle\langle j|)\\ &= \sum_k \lambda_k(\sum_{i=1}^N x_k^{(i)}|ii\rangle)(\sum_{i=1}^N x_k^{(i)}|ii\rangle)^\dagger = \sum_k \lambda_k \operatorname{vec}(X_k)\operatorname{vec}(X_k)^\dagger,\end{aligned}$$

where $\operatorname{vec}(X_k) = \sum_{i=1}^N x_k^{(i)}|ii\rangle \in \mathcal{H}\otimes\mathcal{H}$. Moreover, it is easy to show that $\operatorname{vec}(X_m)^\dagger\operatorname{vec}(X_n) = \delta_{mn}$, thus $\widetilde{\rho}$ is a state on $\mathcal{H}\otimes\mathcal{H}$. That $\mathsf{S}(\widetilde{\rho}) = \mathsf{S}(\rho)$ is clear.

Let $\Lambda \in \mathsf{T}(\mathcal{H})$ be stochastic. If Λ has two Kraus representations $\Lambda = \sum_{p=1}^{d_1} Ad_{S_p} = \sum_{q=1}^{d_2} Ad_{T_q}$, $\rho \in \mathsf{D}(\mathcal{H})$, take two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\dim\mathcal{H}_1 = d_1$, $\dim\mathcal{H}_2 = d_2$, $\{|m\rangle\}$ and $\{|\mu\rangle\}$ are the bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Define

$$\gamma_1(\Lambda) = \sum_{m,n=1}^{d_1}\operatorname{Tr}(S_m\rho S_n^\dagger)|m\rangle\langle n|,\quad \gamma_2(\Lambda) = \sum_{\mu,\nu=1}^{d_2}\operatorname{Tr}(T_\mu\rho T_\nu^\dagger)|\mu\rangle\langle\nu|,$$

then γ_1 and γ_2 are the states on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $\mathsf{S}(\gamma_1(\Lambda)) = \mathsf{S}(\gamma_2(\Lambda))$.

In fact, without loss of generality, we may assume $d_1 = d_2 = d$. Then there exists a $d\times d$ unitary matrix $U = [u_{m\mu}]$ such that for each $1 \leq m \leq d$, $S_m = \sum_{\mu=1}^d u_{m\mu}T_\mu$. Thus

$$\begin{aligned}\sum_{m,n=1}^d \operatorname{Tr}(S_m\rho S_n^\dagger)|m\rangle\langle n| &= \sum_{m,n=1}^d \operatorname{Tr}((\sum_{\mu=1}^d u_{m\mu}T_\mu)\rho(\sum_{\mu=1}^d u_{n\mu}T_\mu)^\dagger)|m\rangle\langle n|\\ &= U\left[\sum_{\mu,\nu=1}^d \operatorname{Tr}(T_\mu\rho T_\nu^\dagger)|m\rangle\langle n|\right]U^\dagger.\end{aligned}$$

Let $V:\mathcal{H}_1\longrightarrow\mathcal{H}_2$ be a unitary operator such that $V|m\rangle=|\mu\rangle$. Then

$$\sum_{m,n=1}^{d}\mathrm{Tr}(S_m\rho S_n^\dagger)|m\rangle\langle n| = UV\left[\sum_{\mu,\nu=1}^{d}\mathrm{Tr}(T_\mu\rho T_\nu^\dagger)|\mu\rangle\langle\nu|\right]V^\dagger U^\dagger,$$

which implied that γ_1 and γ_2 are unitary equivalent and thus the conclusion follows (see [11]).

For each stochastic $\Lambda\in\mathsf{T}(\mathcal{H})$ and $\rho\in\mathsf{D}(\mathcal{H})$, we denote $\mathsf{S}(\rho;\Lambda)$ by $\mathsf{S}(\gamma_1(\Lambda))$, it follows from the above discussion that $\mathsf{S}(\rho;\Lambda)$ is well-defined [10]. Moreover, it is easy to see that if $\rho=\frac{1}{N}\mathbb{1}$, then $\mathsf{S}(\rho;\Lambda)=\mathsf{S}(\Lambda)$ (see [12]).

It follows from above that if $\Phi,\Psi\in T(\mathcal{H})$ are two bi-stochastic quantum operations, $\Phi=\sum_{m=1}^{N^2}Ad_{S_m}$ and $\Psi=\sum_{\mu=1}^{N^2}Ad_{T_\mu}$ are their canonical representations, respectively. Take a N^2 dimensional complex Hilbert space $\mathcal{H}_0$, for each $\rho\in\mathsf{D}(\mathcal{H})$, we define

$$\gamma(\Phi\circ\Psi)=\sum_{m,n,\mu,\nu=1}^{N^2}\mathrm{Tr}(S_mT_\mu\rho(S_nT_\nu)^\dagger)|m\mu\rangle\langle n\nu|,$$

then $\gamma(\Phi\circ\Psi)$ is a state on $\mathcal{H}_0\otimes\mathcal{H}_0$, and when $\rho=\frac{1}{N}\mathbb{1}$, $\mathsf{S}(\gamma(\Phi\circ\Psi))=\mathsf{S}(\Phi\circ\Psi)$, that is, $\mathsf{S}(\rho,\Phi\circ\Psi)=\mathsf{S}(\Phi\circ\Psi)$.

Our mail result in this section is the following:

Theorem 2.1. *Let $\Phi,\Psi\in T(\mathcal{H})$ be two bi-stochastic quantum operations, $\Phi(\rho)=\sum_{m=1}^{N^2}Ad_{S_m}$ and $\Psi=\sum_{\mu=1}^{N^2}Ad_{T_\mu}$ be their canonical representations, respectively. Then $S(\Phi\circ\Psi)=S(\Phi)+S(\Psi)$ if and only if $\mathrm{Tr}(S_mT_\mu(S_nT_\nu)^\dagger)=\frac{1}{N}\mathrm{Tr}(S_mS_n^\dagger)\,\mathrm{Tr}(T_\mu T_\nu^\dagger)$; i.e., $\langle S_nT_\nu,S_mT_\mu\rangle=\frac{1}{N}\langle S_n,S_m\rangle\langle T_\nu,T_\mu\rangle$ for all $m,n,\mu,\nu=1,\ldots,N^2$.*

Proof. The Jamiołkowski isomorphisms of Φ and Ψ are $J(\Phi)=\sum_{m=1}^{N^2}\mathrm{vec}(S_m)\,\mathrm{vec}(S_m)^\dagger$ and $J(\Psi)=\sum_{\mu=1}^{N^2}\mathrm{vec}(T_\mu)\,\mathrm{vec}(T_\mu)^\dagger$, respectively, where $\langle\mathrm{vec}(S_m),\mathrm{vec}(S_n)\rangle=s_m\delta_{mn}$ and $\langle\mathrm{vec}(T_\mu),\mathrm{vec}(T_\nu)\rangle=t_m\delta_{\mu\nu}$. For each $\rho\in\mathsf{D}(\mathcal{H})$, let

$$\gamma(\Phi\circ\Psi)=\sum_{m,n,\mu,\nu=1}^{N^2}\mathrm{Tr}(S_mT_\mu\rho(S_nT_\nu)^\dagger)|m\mu\rangle\langle n\nu|=\sum_{m,n,\mu,\nu=1}^{N^2}\mathrm{Tr}(S_mT_\mu\rho(S_nT_\nu)^\dagger)|m\rangle\langle n|\otimes|\mu\rangle\langle\nu|.$$

Then we have

$$\begin{aligned}\gamma(\Psi)&=\sum_{\mu,\nu=1}^{N^2}\mathrm{Tr}(T_\mu\rho T_\nu^\dagger)|\mu\rangle\langle\nu|=\mathrm{Tr}_1(\gamma(\Phi\circ\Psi)),\\ \gamma(\Phi)&=\sum_{m,n=1}^{N^2}\mathrm{Tr}(S_m\rho S_n^\dagger))|m\rangle\langle n|=\mathrm{Tr}_2(\gamma(\Phi\circ\Psi)).\end{aligned}$$

Note that when $\rho = \frac{1}{N}\mathbb{1}$, $\mathsf{S}(\gamma(\Phi \circ \Psi)) = \mathsf{S}(\Phi \circ \Psi)$, $\mathsf{S}(\gamma(\Psi)) = \mathsf{S}(\Psi)$ and $\mathsf{S}(\gamma(\Phi)) = \mathsf{S}(\Phi)$. Thus, we have

$$\begin{aligned}
\mathsf{S}(\Phi \circ \Psi) = \mathsf{S}(\Phi) + \mathsf{S}(\Psi) \quad &\Leftrightarrow \quad \mathsf{S}(\gamma(\Phi)) + \mathsf{S}(\gamma(\Psi)) = \mathsf{S}(\gamma(\Phi \circ \Psi)) \\
&\Leftrightarrow \quad \gamma(\Phi \circ \Psi) = \gamma(\Phi) \otimes \gamma(\Psi) \\
&\Leftrightarrow \quad \mathrm{Tr}(S_m T_\mu (S_n T_\nu)^\dagger) = \frac{1}{N}\mathrm{Tr}(S_m S_n^\dagger)\,\mathrm{Tr}(T_\mu T_\nu^\dagger) \\
&\quad\quad = \frac{s_m t_\mu}{N}\delta_{mn}\delta_{\mu\nu} (\forall m, n, \mu, \nu = 1, \ldots, N^2).
\end{aligned}$$

□

3. Bi-orthogonal Decomposition and Strong Dynamical Additivity

In order to study the strong dynamical additivity, we need the following bi-orthogonality and the bi-orthogonal decomposition of quantum operations.

Let $\Phi, \Psi \in \mathsf{T}(\mathcal{H})$ be CP maps. If their Jamiołkowski isomorphisms $J(\Psi)$ and $J(\Psi)$ are bi-orthogonal, then Φ and Ψ are said to be *bi-orthogonal*. If $J(\Phi)$ can be represented as a sum $\sum_k D_k$ of pairwise bi-orthogonal positive semi-definite operator D_k, then we say that Φ has a *bi-orthogonal decomposition*.

If $\Phi = \sum_\mu Ad_{M_\mu}$, $\Psi = \sum_\nu Ad_{N_\nu}$, then Φ and Ψ are bi-orthogonal if and only if $M_\mu^\dagger N_\nu = 0$ and $M_\mu N_\nu^\dagger = 0$ for all μ and ν if and only if $\Phi \circ \Psi^\dagger = 0$ and $\Phi^\dagger \circ \Psi = 0$ if and only if $\Psi \circ \Phi^\dagger = 0$ and $\Psi^\dagger \circ \Phi = 0$.

In fact, note that $J(\Phi) = \sum_\mu \mathrm{vec}(M_\mu)\,\mathrm{vec}(M_\mu)^\dagger$, $J(\Psi) = \sum_\nu \mathrm{vec}(N_\nu)\,\mathrm{vec}(N_\nu)^\dagger$, it follows from

$$\begin{aligned}
\mathrm{Tr}_2(J(\Phi))\,\mathrm{Tr}_2(J(\Psi)) &= \left\{\sum_\mu M_\mu {M_\mu}^\dagger\right\}\left\{\sum_\nu N_\nu {N_\nu}^\dagger\right\} = \sum_{\mu,\nu} M_\mu {M_\mu}^\dagger N_\nu {N_\nu}^\dagger = 0, \\
\mathrm{Tr}_1(J(\Phi))\,\mathrm{Tr}_1(J(\Psi)) &= \left\{\sum_\mu [{M_\mu}^\dagger M_\mu]^\mathsf{T}\right\}\left\{\sum_\nu [{N_\nu}^\dagger N_\nu]^\mathsf{T}\right\} = \sum_{\mu,\nu} [{M_\mu}^\dagger M_\mu]^\mathsf{T}[{N_\nu}^\dagger N_\nu]^\mathsf{T} = 0
\end{aligned}$$

that both $J(\Phi)$ and $J(\Psi)$ are bi-orthogonal if and only if $M_\mu {M_\mu}^\dagger N_\nu {N_\nu}^\dagger = 0$ and ${M_\mu}^\dagger M_\mu {N_\nu}^\dagger N_\nu = 0$ for all μ and ν if and only if $M_\mu^\dagger N_\nu = 0$ and $M_\mu N_\nu^\dagger = 0$ for all μ and ν.

Moreover, if $J(\Phi)$ can be represented as a sum $\sum_k D_k$ of pairwise bi-orthogonal positive semi-definite operators, now, we decompose each D_k by the spectral decomposition theorem as

$$D_k = \sum_i d_k^{(i)}\,\mathrm{vec}(\widetilde{M}_k^{(i)})\,\mathrm{vec}(\widetilde{M}_k^{(i)})^\dagger = \sum_i \mathrm{vec}(M_k^{(i)})\,\mathrm{vec}(M_k^{(i)})^\dagger,$$

where $M_k^{(i)} \in \mathbf{L}(\mathcal{H})$, $\mathrm{vec}(M_k^{(i)}) = \sqrt{d_k^{(i)}}\,\mathrm{vec}(\widetilde{M}_k^{(i)})$ and $\langle M_k^{(i)}, M_k^{(j)}\rangle = d_k^{(i)}\delta_{ij}$, then $\Phi_k = \sum_i Ad_{M_k^{(i)}}$ is obtained from $J(\Phi_k) = D_k$. Since $\mathrm{Tr}_2 D_k = \sum_i M_k^{(i)} {M_k^{(i)}}^\dagger$ and $\mathrm{Tr}_1 D_k = \sum_i [{M_k^{(i)}}^\dagger M_k^{(i)}]^\mathsf{T}$, it follows form the bi-orthogonality of $\{D_k\}$ that $M_s^{(i)\dagger} M_t^{(j)} = 0$ and $M_s^{(i)} M_t^{(j)\dagger} = 0$ for any $s \neq t$ and all sub-indices i, j. This implies that $\Phi_m{}^\dagger \circ \Phi_n = 0$ and $\Phi_m \circ \Phi_n{}^\dagger = 0$ if $m \neq n$.

Hence Φ has a bi-orthogonal decomposition if and only if $\Phi = \sum_k \Phi_k$, where $\{\Phi_k\}$ is a collection of CP maps in $\mathsf{T}(\mathcal{H})$ and $\Phi_m{}^\dagger \circ \Phi_n = 0$ and $\Phi_m \circ \Phi_n{}^\dagger = 0$ for all $m \neq n$.

By Proposition 1 in [13], it follows from above that

(i) Let $\Phi_i, \Psi_i \in \mathsf{T}(\mathcal{H})$ be all CP maps, $i = 1, 2$, Φ_1 and Φ_2 be bi-orthogonal, Ψ_1 and Ψ_2 be bi-orthogonal. Then for any CP map $\Lambda \in \mathsf{T}(\mathcal{H})$, $\Phi_1 \circ \Lambda \circ \Psi_1$ and $\Phi_2 \circ \Lambda \circ \Psi_2$ are also bi-orthogonal.

(ii) If $\Phi, \Psi \in \mathsf{T}(\mathcal{H})$ are CP and bi-orthogonal, then for any positive semi-definite operators $X, Y \in \mathbf{L}(\mathcal{H})$, $\Phi(X)$ and $\Psi(Y)$ are orthogonal.

Our main result in this section is the following:

Theorem 3.1. *Assume that $\Phi, \Lambda, \Psi \in \mathsf{T}(\mathcal{H})$ are all bi-stochastic, and the following conditions hold:*

(i) $\mathcal{H} = \bigoplus_{k=1}^K \mathcal{H}_k^L \otimes \mathcal{H}_k^R$, where $\dim \mathcal{H}_k^L = d_k^L, \dim \mathcal{H}_k^R = d_k^R$ and $\sum_{k=1}^K d_k^L d_k^R = N$;

(ii) $\Phi = \bigoplus_{k=1}^K \Phi_k^L \otimes Ad_{U_k^R}, \Lambda = \bigoplus_{k=1}^K \Lambda_k^L \otimes \Lambda_k^R$, and $\Psi = \bigoplus_{k=1}^K Ad_{V_k^L} \otimes \Psi_k^R$, that is, $\Phi|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = \Phi_k^L \otimes Ad_{U_k^R}, \Psi|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = Ad_{V_k^L} \otimes \Psi_k^R$, and $\Lambda|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = \Lambda_k^L \otimes \Lambda_k^R$, $\Phi_k^L, \Lambda_k^L \in \mathsf{T}(\mathcal{H}_k^L)$ are bi-stochastic, $V_k^L \in \mathbf{L}(\mathcal{H}_k^L)$ are unitary operators, $U_k^R \in \mathbf{L}(\mathcal{H}_k^R)$ are unitary operators and $\Psi_k^R, \Lambda_k^R \in \mathsf{T}(\mathcal{H}_k^R)$ are bi-stochastic.

Then we have the following strong dynamical additivity, that is

$$S(\Phi \circ \Lambda) + S(\Lambda \circ \Psi) = S(\Lambda) + S(\Phi \circ \Lambda \circ \Psi).$$

Proof. Since

$$\Phi \circ \Lambda \circ \Psi = \sum_{k=1}^K \Phi_k^L \circ \Lambda_k^L \circ Ad_{V_k^L} \otimes Ad_{U_k^R} \circ \Lambda_k^R \circ \Psi_k^R$$

is a bi-orthogonal decomposition of $\Phi \circ \Lambda \circ \Psi$, so we have

$$\rho(\Phi \circ \Lambda \circ \Psi) = \sum_{k=1}^K \lambda_k \rho(\Phi_k^L \circ \Lambda_k^L \circ Ad_{V_k^L}) \otimes \rho(Ad_{U_k^R} \circ \Lambda_k^R \circ \Psi_k^R),$$

where $\lambda_k = \frac{1}{N} d_k^L d_k^R$ for each k and $\sum_{k=1}^K \lambda_k = 1$. Thus,

$$\begin{aligned} \mathsf{S}(\Phi \circ \Lambda \circ \Psi) &= \mathsf{H}(\lambda) + \sum_{k=1}^K \lambda_k \mathsf{S}(\Phi_k^L \circ \Lambda_k^L \circ Ad_{V_k^L}) + \sum_{k=1}^K \lambda_k \mathsf{S}(Ad_{U_k^R} \circ \Lambda_k^R \circ \Psi_k^R) \\ &= \mathsf{H}(\lambda) + \sum_{k=1}^K \lambda_k \mathsf{S}(\Phi_k^L \circ \Lambda_k^L) + \sum_{k=1}^K \lambda_k \mathsf{S}(\Lambda_k^R \circ \Psi_k^R). \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathsf{S}(\Phi\circ\Lambda) &= \mathsf{H}(\lambda) + \sum_{k=1}^{K}\lambda_k \mathsf{S}(\Phi_k^L\circ\Lambda_k^L) + \sum_{k=1}^{K}\lambda_k \mathsf{S}(\Lambda_k^R),\\
\mathsf{S}(\Lambda\circ\Psi) &= \mathsf{H}(\lambda) + \sum_{k=1}^{K}\lambda_k \mathsf{S}(\Lambda_k^L) + \sum_{k=1}^{K}\lambda_k \mathsf{S}(\Lambda_k^R\circ\Psi_k^R),\\
\mathsf{S}(\Lambda) &= \mathsf{H}(\lambda) + \sum_{k=1}^{K}\lambda_k \mathsf{S}(\Lambda_k^L) + \sum_{k=1}^{K}\lambda_k \mathsf{S}(\Lambda_k^R),
\end{aligned}$$

where $\mathsf{H}(\lambda) = -\sum_{k=1}^{K}\lambda_k\log_2\lambda_k$ is the *Shannon entropy* of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K)$. It follows from these equalities that $\mathsf{S}(\Phi\circ\Lambda) + \mathsf{S}(\Lambda\circ\Psi) = \mathsf{S}(\Lambda) + \mathsf{S}(\Phi\circ\Lambda\circ\Psi)$. □

4. Concluding Remarks

If the entropy $\mathsf{S}(\Phi)$ of a stochastic quantum operation Φ is used to describe the capability of inducing noise induced by Φ, then Theorem 2.1 showed that the capability of inducing noise by the composite operation $\Phi\circ\Psi$ can be separated into two parts induced by operations Φ and Ψ if and only if the conditions of Theorem 2.1 are satisfied. In general, the dynamical subadditivity inequality is strictly, for example, let dim $(\mathcal{H}) = 3$, P is a project operator and dim $(P(\mathcal{H})) = 2$, $\Phi = \Psi = Ad_P + Ad_{\mathbb{1}-P}$, then the conditions of Theorem 2.1 are not satisfied, so $\mathsf{S}(\Phi\circ\Psi) < \mathsf{S}(\Phi) + \mathsf{S}(\Psi)$. Moreover, ones can use the nonnegative quantity $\mathsf{S}(\Phi) + \mathsf{S}(\Psi) - \mathsf{S}(\Phi\circ\Psi)$ to express some correlation between Φ and Ψ, we will discuss this problem later.

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