On Conjectures of Classical and Quantum Correlations in Bipartite States

Lin Zhang and Junde Wu
Department of Mathematics, Zhejiang University,
Hangzhou, 310027, P. R. China
E-mail: linyz@zju.edu.cn; wjd@zju.edu.cn

Professor S. Luo in [Phys. Rev. A **82**, 052122(2010)] proposed two conjectures on the classical correlation and quantum correlation in a bipartite state ρ^{AB}, respectively. In this paper, we prove the conjecture on the classical correlation completely. Moreover, we show that $Q(\rho^{AB}) \leqslant \mathsf{S}(\rho^B)$ is always valid, and the conjecture on quantum correlation is true if $\mathsf{S}(\rho^B) \leqslant \mathsf{S}(\rho^A)$ or ρ^{AB} is separable. We obtain also a class of states ρ^{AB} satisfies that $\mathsf{S}(\rho^A) \leqslant \mathsf{S}(\rho^B)$, but $Q(\rho^{AB}) \leqslant \mathsf{S}(\rho^A)$, so the conjecture on quantum correlation is also true for them.

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I. INTRODUCTION

In quantum information theory, each realizable physical set-up that processes states of quantum system is described by a quantum operation [1] which is mathematically represented by a linear, completely positive super-operator from a set of quantum states to another. The information encoded in a given quantum state is quantified by its von Neumann entropy. In general, the decoherence will be induced in the quantum system when the quantum state is acted by a quantum operation. There are few general and quantitative investigation on the decorrelating capabilities of quantum operations although the decoherent effects of quantum operations are popularly realized.

In order to investigate the decorrelating capabilities of quantum operations, Luo [2] suggested that the decorrelating capabilities of quantum operations should be separated into classical and quantum parts, and the decoherence involved should be related to the quantum part. By the duality of quantum operations and quantum states, each quantum operation can be identified with a bipartite state via the well-known Choi-Jamiołkowski isomorphism [3]. Thus the study of the decorrelating capabilities of quantum operations may be transformed into the investigation of correlations of its corresponding Choi-Jamiołkowski bipartite states. In view of this, the total correlations in a bipartite state play an essential role in the study of the decorrelating capabilities of quantum operations. In order to get some finer quantitative results, after the total correlation was separated into classical and quantum parts, two related conjectures were proposed by Luo in [2] with some supporting examples. In this paper, we studied the two conjectures.

II. CLASSICAL AND QUANTUM CORRELATIONS IN BIPARTITE STATES

Let $\mathscr{H}^1$ be a finite dimensional complex Hilbert space. A quantum operation Φ on $\mathscr{H}^1$ is a completely positive linear super-operator defined on the set of the quantum states on $\mathscr{H}^1$. It follows from ([4], Prop. 5.2 and Coro. 5.5) that there exists linear operators $\{M_\mu\}_{\mu=1}^K$ on $\mathscr{H}^1$ such that $\sum_{\mu=1}^K M_\mu^\dagger M_\mu = \mathbb{1}^1$ and for each quantum state ρ on $\mathscr{H}^1$, we have the Kraus representation

$$\Phi(\rho) = \sum_{\mu=1}^K M_\mu \rho M_\mu^\dagger.$$

Moreover, let $\mathscr{H}^2 = \mathbb{C}^K$ and $\{|\mu\rangle\}_{\mu=1}^K$ be the standard orthonormal basis of $\mathscr{H}^2$. If we define $V : \mathscr{H}^1 \longrightarrow \mathscr{H}^1 \otimes \mathscr{H}^2$ by

$$V|\psi\rangle = \sum_{\mu=1}^K M_\mu|\psi\rangle \otimes |\mu\rangle, \quad |\psi\rangle \in \mathscr{H}^1,$$

then V is an isometry and for each quantum state ρ on $\mathscr{H}^1$, we have the Stinespring representation

$$\Phi(\rho) = \mathrm{Tr}_2(V\rho V^\dagger).$$

It is easy to see that

$$V\rho V^\dagger \;=\; \sum_{\mu,\nu} M_\mu \rho M_\nu^\dagger \otimes |\mu\rangle\langle\nu|.$$

On the other hand, note that for each state ρ on $\mathscr{H}^1$, $\mathrm{Tr}_1(V\rho V^\dagger)$ is a state on $\mathscr{H}^2$, thus, the map

$$\widehat{\Phi} : \rho \mapsto \mathrm{Tr}_1(V\rho V^\dagger) = \sum_{\mu,\nu} \mathrm{Tr}(M_\mu \rho M_\nu^\dagger)|\mu\rangle\langle\nu|$$

is a quantum operation from quantum system $\mathscr{H}^1$ to quantum system $\mathscr{H}^2$, we call it *complementary* to Φ.

If we consider $\mathscr{H}^2$ to be the environment and denote the state $\widehat{\Phi}(\rho)$ by $\hat{\sigma}(\Phi,\rho)$, then $\hat{\sigma}(\Phi,\rho)$ is the state of the environment after the interaction and is called a *correlation matrix*. If the initial state ρ is pure, then the von Neumann entropy $\mathsf{S}(\hat{\sigma}(\Phi,\rho)) = -\mathrm{Tr}(\hat{\sigma}(\Phi,\rho)\log_2 \hat{\sigma}(\Phi,\rho))$ of $\hat{\sigma}(\Phi,\rho)$ describes the entropy exchanged between the system and the environment. Therefore, $\mathsf{S}(\hat{\sigma}(\Phi,\rho))$ is called the *exchange entropy*. The relationship among the $\mathsf{S}(\Phi(\rho)), \mathsf{S}(\rho)$, and $\mathsf{S}(\hat{\sigma}(\Phi,\rho))$ is connected by the well-known Lindblad's entropy inequality [5]:

$$|\mathsf{S}(\hat{\sigma}(\Phi,\rho)) - \mathsf{S}(\rho)| \leqslant \mathsf{S}(\Phi(\rho)) \leqslant \mathsf{S}(\hat{\sigma}(\Phi,\rho)) + \mathsf{S}(\rho). \tag{1}$$

It follows from $\sum_{\mu=1}^K M_\mu^\dagger M_\mu = \mathbb{1}^1$ that $\{M_\mu\}_{\mu=1}^K$ describes a measurement which transforms the initial state ρ into one of the output states

$$\rho'_\mu = \frac{1}{q_\mu} M_\mu \rho M_\mu^\dagger$$

with probability $q_\mu = \mathrm{Tr}(M_\mu \rho M_\mu^\dagger)$. Thus, $\{q_\mu, \rho'_\mu\}$ is a quantum ensemble and its *Holevo quantity* is defined by

$$\chi(\{q_\mu, \rho'_\mu\}) = \mathsf{S}(\sum_\mu q_\mu \rho'_\mu) - \sum_\mu q_\mu \mathsf{S}(\rho'_\mu).$$

Let $\mathsf{H}(\{q_\mu\}) = -\sum_{\mu=1}^k q_\mu \log_2 q_\mu$ be the Shannon entropy of the probability distribution $\{q_\mu\}$. Then we have the following nice inequality [6]:

$$\chi(\{q_\mu, \rho'_\mu\}) \leqslant \mathsf{S}(\hat{\sigma}(\Phi,\rho)) \leqslant \mathsf{H}(\{q_\mu\}). \tag{2}$$

Let $\mathscr{H}^R$ and $\mathscr{H}^Q$ be two finite dimensional complex Hilbert spaces. If Φ^Q is a quantum operation on $\mathscr{H}^Q$, then $\mathbb{1}^R \otimes \Phi^Q$ is a quantum operation on $\mathscr{H}^R \otimes \mathscr{H}^Q$, moreover, if ρ^{RQ} is a state on $\mathscr{H}^R \otimes \mathscr{H}^Q$ and $\rho^Q = \mathrm{Tr}_R(\rho^{RQ})$, then we have [7]:

$$\mathsf{S}(\hat{\sigma}(\mathbb{1}^R \otimes \Phi^Q, \rho^{RQ})) = \mathsf{S}(\hat{\sigma}(\Phi^Q, \rho^Q)). \tag{3}$$

Let $\mathscr{H}^A$ and $\mathscr{H}^B$ be two finite dimensional complex Hilbert spaces, ρ^{AB} is a state on $\mathscr{H}^A \otimes \mathscr{H}^B$, $\rho^A = \mathrm{Tr}_B(\rho^{AB}), \rho^B = \mathrm{Tr}_A(\rho^{AB})$. Then the total correlation in ρ^{AB} is usually quantified by the quantum mutual information

$$I(\rho^{AB}) = \mathsf{S}(\rho^A) + \mathsf{S}(\rho^B) - \mathsf{S}(\rho^{AB}).$$

In [2], Professor Luo separated the total correlation $I(\rho^{AB})$ into classical correlation $C(\rho^{AB})$ and quantum correlation $I(\rho^{AB}) - C(\rho^{AB})$, where the classical correlation $C(\rho^{AB})$ was defined by

$$C(\rho^{AB}) = \sup_{\Pi^B} I[\Pi^B(\rho^{AB})],$$

the sup is taken over all von Neumann measurements $\Pi^B = \{\Pi_j^B\}$ on $\mathscr{H}^B$, and

$$\Pi^B(\rho^{AB}) = \sum_j (\mathbb{1}^A \otimes \Pi_j^B)\rho^{AB}(\mathbb{1}^A \otimes \Pi_j^B)$$

is the output state after executing the nonselective measurement $\Pi^B = \{\Pi_j^B\}$; $\mathbb{1}^A$ is the identity operator on $\mathscr{H}^A$.

The quantum correlation $Q(\rho^{AB})$ was also called quantum discord [8].

In [2], Professor Luo proposed the following conjectures:

$$\begin{aligned} C(\rho^{AB}) &\leqslant \min\{\mathsf{S}(\rho^A), \mathsf{S}(\rho^B)\}, \qquad (I) \\ Q(\rho^{AB}) &\leqslant \min\{\mathsf{S}(\rho^A), \mathsf{S}(\rho^B)\}. \qquad (II) \end{aligned}$$

In this paper, we prove the conjecture (I) completely. Moreover, we show that $Q(\rho^{AB}) \leqslant \mathsf{S}(\rho^B)$ is always valid, and the conjecture (II) is true if $\mathsf{S}(\rho^B) \leqslant \mathsf{S}(\rho^A)$ or ρ^{AB} is separable. We obtain also a class of states ρ^{AB} satisfies that $\mathsf{S}(\rho^A) \leqslant \mathsf{S}(\rho^B)$, but $Q(\rho^{AB}) \leqslant \mathsf{S}(\rho^A)$, so the conjecture on quantum correlation is also true for them.

III. THE PROOF OF THE CONJECTURE

Our main results are the following:

Theorem III.1. *Let ρ^{AB} be a quantum state on $\mathscr{H}^A \otimes \mathscr{H}^B$. Then we have*

(i) $C(\rho^{AB}) \leqslant \min\{\mathsf{S}(\rho^A), \mathsf{S}(\rho^B)\}$,

(ii) $Q(\rho^{AB}) \leqslant \mathsf{S}(\rho^B)$, *and* $Q(\rho^{AB}) \leqslant \min\{\mathsf{S}(\rho^A), \mathsf{S}(\rho^B)\}$ *whenever* $\mathsf{S}(\rho^B) \leqslant \mathsf{S}(\rho^A)$ *or* ρ^{AB} *is separable.*

Proof. (i). Let $\{|\psi_j^B\rangle\}_{j=1}^k$ be a orthonormal basis of $\mathscr{H}^B$ and $\Pi_j^B = |\psi_j^B\rangle\langle\psi_j^B|$. Then $\mathrm{Tr}((\mathbb{1}^A \otimes \Pi_j^B)\rho^{AB}(\mathbb{1}^A \otimes \Pi_j^B)) = \langle\psi_j^B|\rho^B|\psi_j^B\rangle$. If we denote $\langle\psi_j^B|\rho^B|\psi_j^B\rangle$ by p_j, then $p_j \geqslant 0$ and $\sum_j p_j = 1$. Without loss of generality, we assume that all $p_j > 0$. Now, we define

$$\rho_j^A = \frac{(\mathbb{1}^A \otimes \langle\psi_j^B|)\rho^{AB}(\mathbb{1}^A \otimes |\psi_j^B\rangle)}{p_j},$$

then ρ_j^A is a state on $\mathscr{H}^A \otimes \mathscr{H}^B$ and

$$\Pi^B(\rho^{AB}) = \sum_j p_j \rho_j^A \otimes \Pi_j^B,$$

$$\Pi^B(\rho^B) = \sum_j \Pi_j^B \rho^B \Pi_j^B = \sum_j p_j \Pi_j^B,$$

$$\rho^A = \sum_j p_j \rho_j^A.$$

Thus,

$$\begin{aligned} \mathsf{S}(\Pi^B(\rho^{AB})) &= \mathsf{H}(\{p_j\}) + \sum_j p_j \mathsf{S}(\rho_j^A), \\ \mathsf{S}(\Pi^B(\rho^B)) &= \mathsf{H}(\{p_j\}), \end{aligned}$$

and

$$\begin{aligned} I[\Pi^B(\rho^{AB})] &= \mathsf{S}(\rho^A) + \mathsf{S}(\Pi^B(\rho^B)) - \mathsf{S}(\Pi^B(\rho^{AB})) \\ &= \mathsf{S}(\rho^A) - \sum_j p_j \mathsf{S}(\rho_j^A) \\ &= \chi(\{p_j, \rho_j^A\}). \end{aligned}$$

Note that $\sum_j p_j \mathsf{S}(\rho_j^A) \geqslant 0$. Hence $I[\Pi^B(\rho^{AB})] \leqslant \mathsf{S}(\rho^A)$. Thus $C(\rho^{AB}) = \sup_{\Pi^B} I[\Pi^B(\rho^{AB})] \leqslant \mathsf{S}(\rho^A)$.

On the other hand, it follows from $C(\rho^{AB}) = \sup_{\Pi^B} I[\Pi^B(\rho^{AB})]$ and $I[\Pi^B(\rho^{AB})] = \chi(\{p_j, \rho_j^A\}$ that in order to prove $C(\rho^{AB}) \leqslant \mathsf{S}(\rho^B)$, we only need to prove $\chi(\{p_j, \rho_j^A\}) \leqslant \mathsf{S}(\rho^B)$. Note that the quantum ensemble $\{p_j, \rho_j^A\}$ is obtained from the quantum operation of taking partial trace over $\mathcal{H}^B$ from the quantum state ρ^{AB}, this inspired us to define the following quantum operation Ψ on the quantum system $\mathscr{H}^A \otimes \mathscr{H}^B$:

Let $|\omega^B\rangle \in \mathscr{H}^B$ be a fixed unit vector, for each quantum state σ^{AB} on $\mathscr{H}^A \otimes \mathscr{H}^B$,

$$\begin{aligned} \Psi(\sigma^{AB}) &= \sum_j (\mathbb{1}^A \otimes |\omega^B\rangle\langle\psi_j^B|)\sigma^{AB}(\mathbb{1}^A \otimes |\psi_j^B\rangle\langle\omega^B|) \\ &= \mathrm{Tr}_B(\sigma^{AB}) \otimes |\omega^B\rangle\langle\omega^B|. \end{aligned}$$

Let $\mathscr{H}^C = \mathbb{C}^k$ and $\{|i\rangle\}_{i=1}^k$ be the standard orthonormal basis of $\mathscr{H}^C$. Then the correlation matrix $\hat{\sigma}(\Psi, \rho^{AB})$ is given by

$$\begin{aligned} \hat{\sigma}(\Psi, \rho^{AB}) &= \sum_{i,j} \mathrm{Tr}((\mathbb{1}^A \otimes |\omega^B\rangle\langle\psi_i^B|)\rho^{AB}(\mathbb{1}^A \otimes |\psi_j^B\rangle\langle\omega^B|))|i\rangle\langle j| \\ &= \sum_{i,j} \langle\psi_i^B|\rho^B|\psi_j^B\rangle|i\rangle\langle j|, \end{aligned}$$

If we define $W = \sum_j |j\rangle\langle\psi_j^B|$, then $W^\dagger W = \mathbb{1}^B, WW^\dagger = \mathbb{1}^C$, that is, W is an unitary operator from $\mathcal{H}^B$ to $\mathcal{H}^C$. It follows from $\hat{\sigma}(\Psi, \rho^{AB}) = W\rho^B W^\dagger$ that $\mathsf{S}(\hat{\sigma}(\Psi, \rho^{AB})) = \mathsf{S}(\rho^B)$. Note that the quantum ensemble $\{p_j, \rho_j^A \otimes |\omega^B\rangle\langle\omega^B|\}$ can be obtained by the quantum operation Ψ and $\chi(\{p_j, \rho_j^A\}) = \chi(\{p_j, \rho_j^A \otimes |\omega^B\rangle\langle\omega^B|\})$. By using the inequality (2) we have

$$\chi(\{p_j, \rho_j^A\}) \;=\; \chi(\{p_j, \rho_j^A \otimes |\omega^B\rangle\langle\omega^B|\}) \leqslant \mathsf{S}(\hat{\sigma}(\Psi, \rho^{AB})) = \mathsf{S}(\rho^B).$$

Thus, we have proved $C(\rho^{AB}) \leqslant \min\{\mathsf{S}(\rho^A), \mathsf{S}(\rho^B)\}$.

(ii). Note that equality (3) shows that $\mathsf{S}(\hat{\sigma}(\Pi^B, \rho^{AB})) = \mathsf{S}(\hat{\sigma}(\Pi^B, \rho^B))$. Hence it follows from inequality (1) that

$$\mathsf{S}(\Pi^B(\rho^{AB})) - \mathsf{S}(\rho^{AB}) \leqslant \mathsf{S}(\hat{\sigma}(\Pi^B, \rho^{AB})) = \mathsf{S}(\hat{\sigma}(\Pi^B, \rho^B)) = \mathsf{H}(\{p_j\}) = \mathsf{S}(\Pi^B(\rho^B)). \tag{4}$$

On the other hand, note that $I(\Pi^B(\rho^{AB}) = \mathsf{S}(\rho^A) + \mathsf{S}(\Pi^B(\rho^B) - \mathsf{S}(\Pi^B(\rho^{AB})$, by the definition of $Q(\rho^{AB})$ and inequality (4) we have

$$Q(\rho^{AB}) = I(\rho^{AB}) - C(\rho^{AB}) \leqslant \mathsf{S}(\Pi^B\rho^{AB}) - \mathsf{S}(\rho^{AB}) - \mathsf{S}(\Pi^B\rho^B) + \mathsf{S}(\rho^B) \leqslant \mathsf{S}(\rho^B).$$

This showed that $Q(\rho^{AB}) \leqslant \mathsf{S}(\rho^B)$.

Clearly, when $\mathsf{S}(\rho^B) \leqslant \mathsf{S}(\rho^A)$, it follows from $Q(\rho^{AB}) \leqslant \mathsf{S}(\rho^B)$ that $Q(\rho^{AB}) \leqslant \min\{\mathsf{S}(\rho^A), \mathsf{S}(\rho^B)\}$.

If ρ^{AB} is a separable state, then $\mathsf{S}(\rho^{AB}) \geqslant \max\{\mathsf{S}(\rho^A), \mathsf{S}(\rho^B)\}$ [9]. Note that $I(\Pi^B(\rho^{AB})) \geqslant 0$, so $\mathsf{S}(\rho^B) - \mathsf{S}(\rho^{AB}) \leqslant I(\Pi^B(\rho^{AB}))$. Thus, we can prove easily that $Q(\rho^{AB}) \leqslant \min\{\mathsf{S}(\rho^A), \mathsf{S}(\rho^B)\}$. The theorem is proved.

□

In what follows, in order to provide a class of states ρ^{AB} satisfies that $\mathsf{S}(\rho^A) \leqslant \mathsf{S}(\rho^B)$ and $Q(\rho^{AB}) \leqslant \mathsf{S}(\rho^A)$, we need the following:

Theorem III.2. *Let $\mathscr{H}^B$ and $\mathscr{H}^C$ be two finite dimensional complex Hilbert spaces, ρ^{BC} be a state on $\mathscr{H}^B \otimes \mathscr{H}^C$, $\rho^B = \mathrm{Tr}_C(\rho^{BC})$, $\rho^C = \mathrm{Tr}_B(\rho^{BC})$. Then $S(\rho^{BC}) = S(\rho^B) - S(\rho^C)$ if and only if*

(i) H^B can be factorized into the form $\mathscr{H}^B = \mathscr{H}^L \otimes \mathscr{H}^R$,

(ii) $\rho^{BC} = \rho^L \otimes |\Psi^{RC}\rangle\langle\Psi^{RC}|$, where $|\Psi^{RC}\rangle \in \mathscr{H}^R \otimes \mathscr{H}^C$.

Proof. ($\Longleftarrow$) It is trivially.
($\Longrightarrow$) Assume that $\mathsf{S}(\rho^{BC}) = \mathsf{S}(\rho^B) - \mathsf{S}(\rho^C)$. The quantum state ρ^{BC} can be purified into a tripartite state $|\Omega^{ABC}\rangle \in \mathscr{H}^A \otimes \mathscr{H}^B \otimes \mathscr{H}^C$, where $\mathscr{H}^A$ is a reference system. If we denote $\rho^{ABC} = |\Omega^{ABC}\rangle\langle\Omega^{ABC}|$, then

$$\mathrm{Tr}_{AB}(\rho^{ABC}) = \rho^C, \mathrm{Tr}_{AC}(\rho^{ABC}) = \rho^B,$$
$$\mathrm{Tr}_C(\rho^{ABC}) = \rho^{AB}, \mathrm{Tr}_A(\rho^{ABC}) = \rho^{BC}.$$

Note that $\mathsf{S}(\rho^{ABC}) = 0$, so $\mathsf{S}(\rho^C) = \mathsf{S}(\rho^{AB})$, thus, we have

$$\mathsf{S}(\rho^{AB}) + \mathsf{S}(\rho^{BC}) = \mathsf{S}(\rho^B) = \mathsf{S}(\rho^B) + \mathsf{S}(\rho^{ABC}),$$

it follows from [10] that

(i) $\mathscr{H}^B$ can be factorized into the form $\mathscr{H}^B = \bigoplus_{k=1}^K \mathscr{H}_k^L \otimes \mathscr{H}_k^R$,

(ii) $\rho^{ABC} = \bigoplus_{k=1}^K \lambda_k \rho_k^{AL} \otimes \rho_k^{RC}$, where ρ_k^{AL} is a state on $\mathscr{H}^A \otimes \mathscr{H}_k^L$, ρ_k^{RC} is a state on $\mathscr{H}_k^R \otimes \mathscr{H}^C$, $\{\lambda_k\}$ is a probability distribution.

That $\mathsf{S}(\rho^{BC}) = \mathsf{S}(\rho^B) - \mathsf{S}(\rho^C)$ implies $\mathsf{S}(\rho^A) + \mathsf{S}(\rho^C) = \mathsf{S}(\rho^{AC})$ is clear, and $\mathsf{S}(\rho^A) + \mathsf{S}(\rho^C) = \mathsf{S}(\rho^{AC})$ if and only if $\rho^{AC} = \rho^A \otimes \rho^C$ holds. By the expression form of $\rho^{ABC} = \bigoplus_{k=1}^K \lambda_k \rho_k^{AL} \otimes \rho_k^{RC}$, we have $\rho^{AC} = \sum_{k=1}^K \lambda_k \rho_k^A \otimes \rho_k^C$. Combining these facts we have $K = 1$, i.e., the statement (i) of the theorem holds. Hence $\rho^{ABC} = \rho^{AL} \otimes \rho^{RC}$, where ρ^{AL} is a state on $\mathscr{H}^A \otimes \mathscr{H}^L$ and ρ^{RC} is a state on $\mathscr{H}^R \otimes \mathscr{H}^C$, it follows from ρ^{ABC} is pure state that both ρ^{AL} and ρ^{RC} are also pure states. Therefore

$$\rho^{BC} = \mathrm{Tr}_A(\rho^{AL}) \otimes \rho^{RC} = \rho^L \otimes |\Psi^{RC}\rangle\langle\Psi^{RC}|.$$

The statement (ii) holds and the theorem is proved. □

Example III.3. Let ρ^{AB} be a bipartite state on $\mathscr{H}^A \otimes \mathscr{H}^B$ such that $\mathsf{S}(\rho^{AB}) = \mathsf{S}(\rho^B) - \mathsf{S}(\rho^A)$. By Theorem III.2, we have $\rho^{AB} = |\Phi^{AL}\rangle\langle\Phi^{AL}| \otimes \rho^R$ for $|\Phi^{AL}\rangle \in \mathscr{H}^A \otimes \mathscr{H}^L$, where ρ^R is a state on $\mathscr{H}^R$ and $\mathscr{H}^B = \mathscr{H}^L \otimes \mathscr{H}^R$. It is easy to show that although $\mathsf{S}(\rho^A) \leqslant \mathsf{S}(\rho^B)$, but $Q(\rho^{AB}) = \mathsf{S}(\rho^A)$, so the conjecture (II) is true for this class of states.

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